CHAPTER

7

Systems of First Order Linear Equations

7.1

2. As in Example 1, let $x_1 = u$ and $x_2 = u'$. Then $x_1' = x_2$ and $x_2' = u'' = 3\sin t - 0.5u' - 2u = -2x_1 - 0.5x_2 + 3\sin t$.

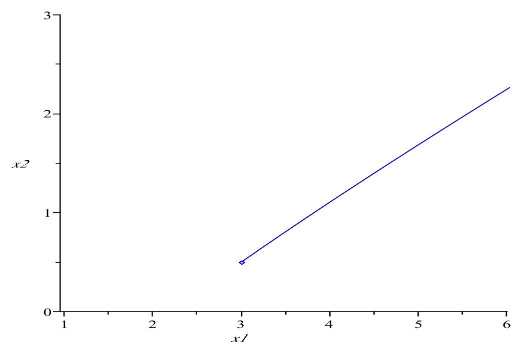
4. In this case let $x_1 = u$, $x_2 = u'$, $x_3 = u''$, and $x_4 = u'''$. The last equation gives $x_4' = x_1$.

6. Let $x_1 = u$ and $x_2 = u'$; then $x_1' = x_2$ is the first of the desired pair of equations. The second equation is obtained by substituting $u'' = x_2'$, $u' = x_2$, and $u = x_1$ in the given differential equation. The initial conditions become $x_1(0) = u_0$, $x_2(0) = u_0'$.

8.(a) We follow the steps outlined in Problem 7. Solving the first equation for x_2 gives $x_2 = (3/2)x_1 - x_1'/2$, substituting this into the second differential equation we obtain $(3/2)x_1' - x_1''/2 = 2x_1 - 2(3x_1/2 - x_1'/2)$, i.e. $x_1'' = x_1' + 2x_1$, which is the same as $x_1'' - x_1' - 2x_1 = 0$.

(b) The general solution of the second order differential equation in part (a) is $x_1 = c_1e^{2t} + c_2e^{-t}$. Differentiating this and substituting into the above equation for x_2 yields $x_2 = c_1e^{2t}/2 + 2c_2e^{-t}$. The initial conditions then give $c_1 + c_2 = 3$ and $c_1/2 + 2c_2 = 1/2$. This implies that $c_1 = 11/3$ and $c_2 = -2/3$. Thus $x_1 = (11e^{2t} - 2e^{-t})/3$ and $x_2 = (11e^{2t} - 8e^{-t})/6$.

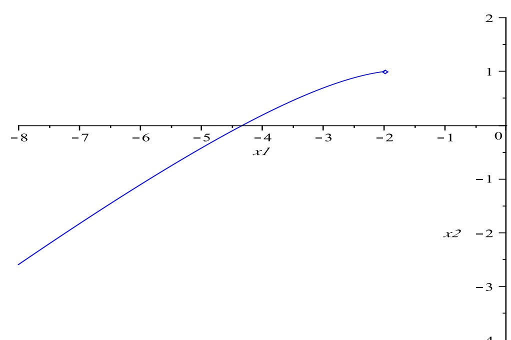
(c)



9.(a) Solving the first equation for x_2 gives $x_2 = (4/3)x_1' - (5/3)x_1$, substituting this into the second differential equation we obtain $(4/3)x_1'' - (5/3)x_1' = (3/4)x_1 + (5/4)((4/3)x_1' - (5/3)x_1)$, i.e. $(4/3)x_1'' = (10/3)x_1' - (4/3)x_1$, which is the same as $x_1'' - (5/2)x_1' + x_1 = 0$.

(b) From part (a), $x_1 = c_1e^{t/2} + c_2e^{2t}$ and $x_2 = -c_1e^{t/2} + c_2e^{2t}$. Using the initial conditions yields $c_1 = -3/2$ and $c_2 = -1/2$. Thus $x_1 = -(3/2)e^{t/2} - (1/2)e^{2t}$ and $x_2 = (3/2)e^{t/2} - (1/2)e^{2t}$.

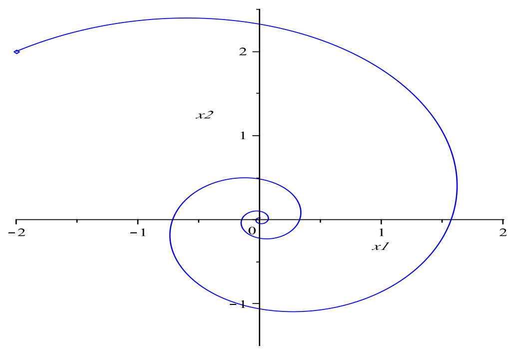
(c)



12.(a) Solving the first equation for x_2, we obtain $x_2 = x_1'/2 + x_1/4$. Substitution into the second equation results in $x_1''/2 + x_1'/4 = -2\,x_1 - (x_1'/2 + x_1/4)/2$. Rearranging the terms, the single differential equation for x_1 is $x_1'' + x_1' + (17/4)x_1 = 0$.

(b) The general solution is $x_1(t) = e^{-t/2}\,[c_1\cos 2t + c_2\sin 2t]$. With x_2 given in terms of x_1, it follows that $x_2(t) = e^{-t/2}\,[c_2\cos 2t - c_1\sin 2t]$. Imposing the specified initial conditions, we obtain $c_1 = -2$ and $c_2 = 2$. This implies that $x_1(t) = e^{-t/2}\,[-2\cos 2t + 2\sin 2t]$ and $x_2(t) = e^{-t/2}\,[2\cos 2t + 2\sin 2t]$.

(c)



14. If $a_{12} \neq 0$, then solve the first equation for x_2, obtaining $x_2 = (x_1' - a_{11}x_1 - g_1(t))/a_{12}$. Upon substituting this expression into the second equation, we have a second order linear differential equation for x_1. One initial condition is $x_1(0) = x_1^0$. The second initial condition is $x_2(0) = (x_1'(0) - a_{11}x_1(0) - g_1(0))/a_{12} = x_2^0$. Solving for $x_1'(0)$ gives $x_1'(0) = a_{12}x_2^0 + a_{11}x_1^0 + g_1(0)$. If $a_{12} = 0$, then solve the second equation for x_1 and proceed as above. These results hold when $a_{11}, \ldots, a_{22}$ are functions of t as long as the derivatives exist and $a_{12}(t)$ and $a_{21}(t)$ are not both zero on the interval. The initial conditions will involve $a_{11}(0)$ and $a_{12}(0)$.

20. Let I_1, I_2, I_3 and I_4 be the current through the 1 ohm resistor, 2 ohm resistor, inductor and capacitor, respectively. Assign V_1, V_2, V_3 and V_4 as the respective voltage drops. Based on Kirchhoff's second law, the net voltage drops around each loop satisfy

$$(1)\ V_1 + V_3 + V_4 = 0, \qquad (2)\ V_1 + V_3 + V_2 = 0 \quad \text{and} \quad (3)\ V_4 - V_2 = 0.$$

Applying Kirchhoff's first law to the upper right node, we have

$$(4)\ I_1 - I_3 = 0.$$

Likewise, in the remaining nodes, we have

$$(5)\ I_2 + I_4 - I_1 = 0 \quad \text{and} \quad (6)\ I_3 - I_4 - I_2 = 0.$$

Using the current-voltage relations, we have

$$(7)\ V_1 = R_1 I_1, \qquad (8)\ V_2 = R_2 I_2, \qquad (9)\ L I_3' = V_3, \qquad (10)\ C V_4' = I_4.$$

Using equations (1) and (6) with substitutions from equations (3) and (4) and utilizing the current-voltage relations we obtain the two equations

$$R_1 I_3 + L I_3' + V_4 = 0 \quad \text{and} \quad C V_4' = I_3 - \frac{1}{R_2} V_4.$$

Now set $I_3 = I$ and $V_4 = V$, to obtain the system of equations

$$L I' = -R_1 I - V \quad \text{and} \quad C V' = I - \frac{1}{R_2} V.$$

Finally, using the fact that $R_1 = 1$, $R_2 = 2$, $L = 1$ and $C = 1/2$, we have

$$I' = -I - V \quad \text{and} \quad V' = 2I - V,$$

as claimed.

22.(a) Let $Q_1(t)$ and $Q_2(t)$ be the amount of salt in the respective tanks at time t. Based on conservation of mass, the rate of increase of salt is given by

$$\text{rate of increase} = \text{rate in} - \text{rate out}.$$

For Tank 1, the rate of salt flowing in from Tank 2 is $(Q_2/20) \cdot 1.5 = .075Q_2$ ounces/minute. In addition, salt is flowing in from a separate source at the rate of 1.5 ounces/minute. Therefore, the rate of salt flowing in to Tank 1 is $r_{in} = .075Q_2 + 1.5$. The rate of flow out of Tank 1 is $r_{out} = (Q_1/30) \cdot 3 = 0.1Q_1$ ounces/minute. Therefore,

$$\frac{dQ_1}{dt} = -0.1Q_1 + .075Q_2 + 1.5.$$

Similarly, for Tank 2, salt is flowing in from Tank 1 at the rate of $(Q_1/30) \cdot 3 = 0.1$ oz/min. In addition, salt is flowing in from a separate source at the rate of 3 oz/min. Also, salt is flowing out of Tank 2 at the rate of $4Q_2/20 = .2Q_2$ oz/min. Therefore,

$$\frac{dQ_2}{dt} = 0.1Q_1 - 0.2Q_2 + 3.$$

The initial conditions are $Q_1(0) = 25$ and $Q_2(0) = 15$.

(b) Solve the second equation for $Q_1(t)$ to obtain $Q_1(t) = 10Q_2' + 2Q_2 - 30$. Substitution into the first equation then yields $10Q_2'' + 3Q_2' + Q_2/8 = 9/2$. Equilibrium is the steady state solution, which is $Q_2^E = 8(9/2) = 36$. Substituting this value into the equation for Q_1 yields $Q_1^E = 72 - 30 = 42$. It can be shown that $Q_1(t)$ satisfies the same second order differential equation as $Q_2(t)$ (except with the constant 21/4 on the right side) and thus the exponentials in the solutions for each are the same. Hence each tank approaches the equilibrium solution at the same rate.

(c) Substitute $Q_1 = x_1 + 42$ and $Q_2 = x_2 + 36$ into the equations found in part (a).

7.2

1.(a)

$$2\mathbf{A} + \mathbf{B} = \begin{pmatrix} 2+4 & -4-2 & 0+3 \\ 6-1 & 4+5 & -2+0 \\ -4+6 & 2+1 & 6+2 \end{pmatrix} = \begin{pmatrix} 6 & -6 & 3 \\ 5 & 9 & -2 \\ 2 & 3 & 8 \end{pmatrix}.$$

(b)

$$\mathbf{A} - 4\mathbf{B} = \begin{pmatrix} 1-16 & -2+8 & 0-12 \\ 3+4 & 2-20 & -1+0 \\ -2-24 & 1-4 & 3-8 \end{pmatrix} = \begin{pmatrix} -15 & 6 & -12 \\ 7 & -18 & -1 \\ -26 & -3 & -5 \end{pmatrix}.$$

(c)

$$\mathbf{AB} = \begin{pmatrix} 4+2+0 & -2-10+0 & 3+0+0 \\ 12-2-6 & -6+10-1 & 9+0-2 \\ -8-1+18 & 4+5+3 & -6+0+6 \end{pmatrix} = \begin{pmatrix} 6 & -12 & 3 \\ 4 & 3 & 7 \\ 9 & 12 & 0 \end{pmatrix}.$$

(d)

$$\mathbf{BA} = \begin{pmatrix} 4-6-6 & -8-4+3 & 0+2+9 \\ -1+15+0 & 2+10+0 & 0-5+0 \\ 6+3-4 & -12+2+2 & 0-1+6 \end{pmatrix} = \begin{pmatrix} -8 & -9 & 11 \\ 14 & 12 & -5 \\ 5 & -8 & 5 \end{pmatrix}.$$

6.(a)

$$\mathbf{AB} = \begin{pmatrix} 6 & -5 & -7 \\ 1 & 9 & 1 \\ -1 & -2 & 8 \end{pmatrix} \quad \text{and} \quad \mathbf{BC} = \begin{pmatrix} 5 & 3 & 3 \\ -1 & 7 & 3 \\ 2 & 3 & -2 \end{pmatrix}.$$

so that

$$(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC}) = \begin{pmatrix} 7 & -11 & -3 \\ 11 & 20 & 17 \\ -4 & 3 & -12 \end{pmatrix}.$$

(b)

$$\mathbf{A}+\mathbf{B} = \begin{pmatrix} 3 & -1 & -1 \\ 1 & 5 & 2 \\ -1 & 0 & 5 \end{pmatrix} \quad \text{and} \quad \mathbf{B}+\mathbf{C} = \begin{pmatrix} 4 & 2 & -1 \\ -1 & 5 & 5 \\ 1 & 1 & 1 \end{pmatrix}.$$

so that

$$(\mathbf{A}+\mathbf{B})+\mathbf{C} = \mathbf{A}+(\mathbf{B}+\mathbf{C}) = \begin{pmatrix} 5 & 0 & -1 \\ 2 & 7 & 4 \\ -1 & 1 & 4 \end{pmatrix}.$$

(c)

$$\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC} = \begin{pmatrix} 6 & -8 & -11 \\ 9 & 15 & 6 \\ -5 & -1 & 5 \end{pmatrix}.$$

10. First augment the given matrix by the identity matrix:

$$[\mathbf{A}\,|\,\mathbf{I}] = \begin{pmatrix} 1 & 4 & 1 & 0 \\ -2 & 3 & 0 & 1 \end{pmatrix}.$$

Add 2 times the first row to the second row:

$$\begin{pmatrix} 1 & 4 & 1 & 0 \\ 0 & 11 & 2 & 1 \end{pmatrix}.$$

Multiply the second row by 1/11:

$$\begin{pmatrix} 1 & 4 & 1 & 0 \\ 0 & 1 & 2/11 & 1/11 \end{pmatrix}.$$

Add -4 times the second row to the first row:

$$\begin{pmatrix} 1 & 0 & 3/11 & -4/11 \\ 0 & 1 & 2/11 & 1/11 \end{pmatrix}.$$

Hence

$$\begin{pmatrix} 1 & 4 \\ -2 & 3 \end{pmatrix}^{-1} = \frac{1}{11}\begin{pmatrix} 3 & -4 \\ 2 & 1 \end{pmatrix}.$$

The answer can be checked by multiplying it by the given matrix; the result should be the identity matrix.

12. The augmented matrix is

$$\begin{pmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 4 & 5 & 0 & 1 & 0 \\ 3 & 5 & 6 & 0 & 0 & 1 \end{pmatrix}.$$

Add -2 times the first row to the second row and -3 times the first row to the third row:

$$\begin{pmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & -1 & -2 & 1 & 0 \\ 0 & -1 & -3 & -3 & 0 & 1 \end{pmatrix}.$$

Multiply the second and third rows by -1 and interchange them:

$$\begin{pmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 3 & 3 & 0 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{pmatrix}.$$

Add -3 times the third row to the first and second rows:

$$\begin{pmatrix} 1 & 2 & 0 & -5 & 3 & 0 \\ 0 & 1 & 0 & -3 & 3 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{pmatrix}.$$

Add -2 times the second row to the first row:

$$\begin{pmatrix} 1 & 0 & 0 & 1 & -3 & 2 \\ 0 & 1 & 0 & -3 & 3 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{pmatrix}.$$

Hence

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{pmatrix}.$$

The answer can be checked by multiplying it by the given matrix; the result should be the identity matrix.

14. The augmented matrix is

$$\begin{pmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ -2 & 1 & 8 & 0 & 1 & 0 \\ 1 & -2 & -7 & 0 & 0 & 1 \end{pmatrix}.$$

Add 2 times the first row to the second row and add -1 times the first row to the third row:

$$\begin{pmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 5 & 10 & 2 & 1 & 0 \\ 0 & -4 & -8 & -1 & 0 & 1 \end{pmatrix}.$$

Add 4/5 times the second row to the third row:

$$\begin{pmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 5 & 10 & 2 & 1 & 0 \\ 0 & 0 & 0 & 3/5 & 4/5 & 0 \end{pmatrix}.$$

Since the third row of the left matrix is all zeros, no further reduction can be performed, and the given matrix is singular.

21.(a)

$$\mathbf{A}+3\mathbf{B} = \begin{pmatrix} e^t & 2e^{-t} & e^{2t} \\ 2e^t & e^{-t} & -e^{2t} \\ -e^t & 3e^{-t} & 2e^{2t} \end{pmatrix} + \begin{pmatrix} 6e^t & 3e^{-t} & 9e^{2t} \\ -3e^t & 6e^{-t} & 3e^{2t} \\ 9e^t & -3e^{-t} & -3e^{2t} \end{pmatrix} = \begin{pmatrix} 7e^t & 5e^{-t} & 10e^{2t} \\ -e^t & 7e^{-t} & 2e^{2t} \\ 8e^t & 0 & -e^{2t} \end{pmatrix}.$$

(b) Based on the standard definition of matrix multiplication,

$$\mathbf{AB} = \begin{pmatrix} 2e^{2t}-2+3e^{3t} & 1+4e^{-2t}-e^t & 3e^{3t}+2e^t-e^{4t} \\ 4e^{2t}-1-3e^{3t} & 2+2e^{-2t}+e^t & 6e^{3t}+e^t+e^{4t} \\ -2e^{2t}-3+6e^{3t} & -1+6e^{-2t}-2e^t & -3e^{3t}+3e^t-2e^{4t} \end{pmatrix}.$$

(c)

$$\frac{d\mathbf{A}}{dt} = \begin{pmatrix} e^t & -2e^{-t} & 2e^{2t} \\ 2e^t & -e^{-t} & -2e^{2t} \\ -e^t & -3e^{-t} & 4e^{2t} \end{pmatrix}.$$

(d) Note that

$$\int \mathbf{A}(t)dt = \begin{pmatrix} e^t & -2e^{-t} & e^{2t}/2 \\ 2e^t & -e^{-t} & -e^{2t}/2 \\ -e^t & -3e^{-t} & e^{2t} \end{pmatrix} + C.$$

Therefore

$$\int_0^1 \mathbf{A}(t)dt = \begin{pmatrix} e & -2e^{-1} & e^2/2 \\ 2e & -e^{-1} & -e^2/2 \\ -e & -3e^{-1} & e^2 \end{pmatrix} - \begin{pmatrix} 1 & -2 & 1/2 \\ 2 & -1 & -1/2 \\ -1 & -3 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} e-1 & 2-2e^{-1} & e^2/2-1/2 \\ 2e-2 & 1-e^{-1} & 1/2-e^2/2 \\ 1-e & 3-3e^{-1} & e^2-1 \end{pmatrix}.$$

The result can also be written as

$$(e-1)\begin{pmatrix} 1 & 2/e & \frac{1}{2}(e+1) \\ 2 & 1/e & -\frac{1}{2}(e+1) \\ -1 & 3/e & e+1 \end{pmatrix}.$$

22. We compute:

$$\mathbf{x}' = \begin{pmatrix} 4 \\ 2 \end{pmatrix} 2e^{2t} = \begin{pmatrix} 8 \\ 4 \end{pmatrix} e^{2t};$$

also,

$$\begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix} e^{2t} = \begin{pmatrix} 8 \\ 4 \end{pmatrix} e^{2t}.$$

25. Differentiation, elementwise, results in

$$\mathbf{\Psi}' = \begin{pmatrix} -3e^{-3t} & 2e^{2t} \\ 12e^{-3t} & 2e^{2t} \end{pmatrix}.$$

On the other hand,

$$\begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \mathbf{\Psi} = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} e^{-3t} & e^{2t} \\ -4e^{-3t} & e^{2t} \end{pmatrix} = \begin{pmatrix} -3e^{-3t} & 2e^{2t} \\ 12e^{-3t} & 2e^{2t} \end{pmatrix}.$$

7.3

1. The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 3 & 1 & 1 & 1 \\ -1 & 1 & 2 & 2 \end{array}\right).$$

Adding -3 times the first row to the second row and adding the first row to the third row results in

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 1 & 1 & 2 \end{array}\right).$$

Adding -1 times the second row to the third row results in

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & -3 & 1 \end{array}\right).$$

The third row is equivalent to $-3x_3 = 1$ or $x_3 = -1/3$. Likewise the second row is equivalent to $x_2 + 4x_3 = 1$, so $x_2 = 7/3$. Finally, from the first row, $x_1 - x_3 = 0$, so $x_1 = -1/3$. The answer can be checked by substituting into the original equations.

2. The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & 1 & 1 & 1 \\ 1 & -1 & 2 & 1 \end{array}\right).$$

Adding -2 times the first row to the second row and adding -1 times the first row to the third row results in

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -3 & 3 & -1 \\ 0 & -3 & 3 & 0 \end{array}\right).$$

Adding -1 times the second row to the third row results in

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -3 & 3 & -1 \\ 0 & 0 & 0 & 1 \end{array}\right).$$

This means we need $0x_1 + 0x_2 + 0x_3 = 1$, thus there is no solution.

3. The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 2 & 1 & 1 & 1 \\ 1 & -1 & 2 & -1 \end{array}\right).$$

Adding -2 times the first row to the second row and subtracting the first row from the third row results in

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -3 & 3 & -3 \\ 0 & -3 & 3 & -3 \end{array}\right).$$

Adding -1 times the second row to the third row and then multiplying the second row by $-1/3$ results in

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

We evidently end up with an equivalent system of equations

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 2 \\ x_2 - x_3 &= 1\,. \end{aligned}$$

Since there is no unique solution, let $x_3 = \alpha$, where α is arbitrary. It follows that $x_2 = 1 + \alpha$, and then $x_1 = 2 - 2x_2 + x_3 = 2 - 2(1+\alpha) + \alpha = -\alpha$. Hence all solutions have the form

$$x = \begin{pmatrix} -\alpha \\ 1+\alpha \\ \alpha \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix},$$

where the first vector on the right is a solution of the given nonhomogeneous equation and the second vector is a solution of the related homogeneous equation.

7. Write the given vectors as columns of the matrix

$$\mathbf{X} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

We find that $\det(\mathbf{X}) = 2 \neq 0$, hence the vectors are linearly independent.

9. We wish to solve the system $c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + c_3\mathbf{x}^{(3)} + c_4\mathbf{x}^{(4)} = \mathbf{0}$. Form the augmented matrix and use row reduction:

$$\begin{pmatrix} 1 & -1 & -2 & -3 & | & 0 \\ 2 & 0 & -1 & 0 & | & 0 \\ 2 & 3 & 1 & -1 & | & 0 \\ 3 & 1 & 0 & 3 & | & 0 \end{pmatrix}.$$

Add -2 times the first row to the second, add -2 times the first row to the third, and add -3 times the firts row to the fourth:

$$\begin{pmatrix} 1 & -1 & -2 & -3 & | & 0 \\ 0 & 2 & 3 & 6 & | & 0 \\ 0 & 5 & 5 & 5 & | & 0 \\ 0 & 4 & 6 & 12 & | & 0 \end{pmatrix}.$$

Multiply the second row by $1/2$ and then add -5 times the second row to the third row and add -4 times the second row to the fourth:

$$\begin{pmatrix} 1 & -1 & -2 & -3 & | & 0 \\ 0 & 1 & 3/2 & 3 & | & 0 \\ 0 & 0 & -5/2 & -10 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}.$$

Choose now $c_4 = -1$ for example (any nonzero number is suitable). Then by the third row of the matrix $c_3 = 4$. From the second row we have $c_2 = -(3/2)c_3 - 3c_4 = -3$. From the first row $c_1 = c_2 + 2c_3 + 3c_4 = 2$. Hence the given vectors are linearly dependent and $2\mathbf{x}^{(1)} - 3\mathbf{x}^{(2)} + 4\mathbf{x}^{(3)} - \mathbf{x}^{(4)} = \mathbf{0}$.

13. By inspection, we find that

$$\mathbf{x}^{(1)}(t) - 2\mathbf{x}^{(2)}(t) = \begin{pmatrix} -e^{-t} \\ 0 \end{pmatrix}.$$

Hence $3\,\mathbf{x}^{(1)}(t) - 6\,\mathbf{x}^{(2)}(t) + \mathbf{x}^{(3)}(t) = \mathbf{0}$, and the vectors are linearly dependent.

14. Two vectors are linearly dependent if and only if one is a nonzero scalar multiple of the other. However, there is no nonzero scalar c such that $2\sin t = c\sin t$ and $\sin t = 2c\sin t$ for all $t \in (-\infty, \infty)$. Therefore the vectors are linearly independent.

15. Let $t = t_0$ be a fixed value of t in the interval $0 \le t \le 1$. We can easily check that $1\mathbf{x}^{(1)} - e^{t_0}\mathbf{x}^{(2)} = \mathbf{0}$ and hence the given vectors are linearly dependent at each point of the interval. However, there is clearly no nonzero scalar c such that $e^t = c \cdot 1$ and $te^t = ct$ on the whole interval $0 \le t \le 1$. So the given vectors are linearly independent on $0 \le t \le 1$.

16. The eigenvalues λ and eigenvectors $\mathbf{x}$ satisfy the equation

$$\begin{pmatrix} 5-\lambda & -1 \\ 3 & 1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $(5-\lambda)(1-\lambda)+3=0$, that is, $\lambda^2-6\lambda+8=0$. The eigenvalues are $\lambda_1=2$ and $\lambda_2=4$. The components of the eigenvector $\mathbf{x}^{(1)}$ are solutions of the system

$$\begin{pmatrix}3 & -1\\ 3 & -1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}.$$

The two equations reduce to $3x_1=x_2$. Hence $\mathbf{x}^{(1)}=(1\,,3)^T$, or any constant multiple of this vector. Now setting $\lambda=\lambda_2=4$, we have

$$\begin{pmatrix}1 & -1\\ 3 & -3\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix},$$

with solution given by $\mathbf{x}^{(2)}=(1\,,1)^T$, or a multiple of thereof.

19. Since $\overline{a_{12}}=a_{21}$, the given matrix is Hermitian and we know in advance that its eigenvalues are real. The eigenvalues λ and eigenvectors $\mathbf{x}$ satisfy the equation

$$\begin{pmatrix}1-\lambda & i\\ -i & 1-\lambda\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}.$$

For a nonzero solution, we must have $(1-\lambda)(1-\lambda)-1=0$, that is,

$$\lambda^2-2\lambda=0\,.$$

The eigenvalues are $\lambda_1=0$ and $\lambda_2=2$. For $\lambda_1=0$, the system of equations becomes

$$\begin{pmatrix}1 & i\\ -i & 1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix},$$

which reduces to $x_1+ix_2=0$. A solution vector is given by $\mathbf{x}^{(1)}=(1\,,i)^T$. Substituting $\lambda=\lambda_2=2$, we have

$$\begin{pmatrix}-1 & i\\ -i & -1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}.$$

The equations reduce to $x_1=ix_2$. So a solution vector is given by $\mathbf{x}^{(2)}=(1\,,-i)^T$.

22. The eigensystem is obtained from analysis of the equation

$$\begin{pmatrix}1-\lambda & 0 & 0\\ 2 & 1-\lambda & -2\\ 3 & 2 & 1-\lambda\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}.$$

The characteristic equation of the coefficient matrix is $(1-\lambda)((1-\lambda)^2+4)=0$, with roots $\lambda=1,\,1\pm 2i$. Setting $\lambda=1$, we have

$$\begin{pmatrix}0 & 0 & 0\\ 2 & 0 & -2\\ 3 & 2 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}.$$

This system gives the equations $x_1-x_3=0$ and $3x_1+2x_2=0$. A corresponding solution vector is given by $\mathbf{x}^{(1)}=(2\,,-3\,,2)^T$. Setting $\lambda=1+2i$, the reduced system of equations is $-2ix_1=0$, $2x_1-2ix_2-2x_3=0$ and $3x_1+2x_2-2ix_3=0$, yielding $x_1 = 0$ and $x_3 = -ix_2$. Thus $\mathbf{x}^{(2)} = (0\,,1\,,-i)^T$ is the eigenvector corresponding to $\lambda = 1+2i$. A similar calculation yields $\mathbf{x}^{(3)} = (0\,,1\,,i)^T$ as the eigenvector corresponding to $\lambda = 1-2i$.

25. Since the given matrix is real and symmetric, we know that the eigenvalues are real. Further, even if there are repeated eigenvalues, there will be a full set of three linearly independent eigenvectors. The eigensystem is obtained from analysis of the equation

$$\begin{pmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The characteristic equation of the coefficient matrix is $(3-\lambda)(-\lambda(3-\lambda)-4) - 2(2(3-\lambda)-8) + 4(4+4\lambda) = -\lambda^3 + 6\lambda^2 + 15\lambda + 8 = 0$, with roots $\lambda_1 = -1$, $\lambda_2 = -1$ and $\lambda_3 = 8$. Setting $\lambda = -1$, we have

$$\begin{pmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This system is reduced to the single equation $2x_1 + x_2 + 2x_3 = 0$. Consequently, two of the three variables can be selected arbitrarily and the third is determined by the equation. For example, choosing $x_1 = 1$ and $x_3 = 1$ gives $x_2 = -4$ and choosing $x_1 = 1$ and $x_2 = 0$ gives $x_3 = -1$. These produce two linearly independent eigenvectors corresponding to -1: $\mathbf{x}^{(1)} = (1\,,-4\,,1)^T$ and $\mathbf{x}^{(2)} = (1\,,0\,,-1)^T$. Setting $\lambda = \lambda_3 = 8$, the reduced system of equations is $x_1 - 4\,x_2 + x_3 = 0$ and $2x_2 - x_3 = 0$. A corresponding solution vector is given by $\mathbf{x}^{(3)} = (2\,,1\,,2)^T$.

28. We are given that $\mathbf{Ax} = \mathbf{b}$ has solutions and thus we have $(\mathbf{Ax}, \mathbf{y}) = (\mathbf{b}, \mathbf{y})$. Using $\mathbf{A}^*\mathbf{y} = \mathbf{0}$ and the result of Problem 26, we have $(\mathbf{Ax}, \mathbf{y}) = (\mathbf{x}, \mathbf{A}^*\mathbf{y}) = 0$. Thus $(\mathbf{b}, \mathbf{y}) = 0$. For Example 2, since $\mathbf{A}$ is real,

$$\mathbf{A}^* = \mathbf{A}^T = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 1 & -1 \\ 3 & -2 & 3 \end{pmatrix},$$

and using row reduction, the augmented matrix for $\mathbf{A}^*\mathbf{y} = \mathbf{0}$ becomes

$$\begin{pmatrix} 1 & -1 & 2 & | & 0 \\ 0 & 1 & -3 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}.$$

Thus $\mathbf{y} = c(1, 3, 1)^T$ and hence $(\mathbf{b}, \mathbf{y}) = b_1 + 3b_2 + b_3 = 0$.

7.4

1. Use mathematical induction. It has already been proven that if $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are solutions, then so is $c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}$. Assume that if $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(k)}$ are solutions, then $\mathbf{x} = c_1\mathbf{x}^{(1)} + \ldots + c_k\mathbf{x}^{(k)}$ is a solution. Then use Theorem 7.4.1 to

conclude that $\mathbf{x} + c_{k+1}\mathbf{x}^{(k+1)}$ is also a solution and thus $c_1\mathbf{x}^{(1)} + \ldots + c_{k+1}\mathbf{x}^{(k+1)}$ is a solution if $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(k+1)}$ are solutions.

2.(a) From Eq.(10) we have

$$W = \begin{vmatrix} x_1^{(1)} & x_1^{(2)} \\ x_2^{(1)} & x_2^{(2)} \end{vmatrix} = x_1^{(1)}x_2^{(2)} - x_2^{(1)}x_1^{(2)}.$$

Taking the derivative of these two products yields four terms which can be written as

$$\frac{dW}{dt} = (\frac{dx_1^{(1)}}{dt}x_2^{(2)} - x_2^{(1)}\frac{dx_1^{(2)}}{dt}) + (x_1^{(1)}\frac{dx_2^{(2)}}{dt} - \frac{dx_2^{(1)}}{dt}x_1^{(2)}).$$

The terms in brackets can now be recognized as the respective determinants appearing in the desired result. A similar result was obtained in Problem 20 of Section 4.1.

(b) If $x_1^{(1)}$ is substituted into Eq.(3) we have

$$\frac{dx_1^{(1)}}{dt} = p_{11}x_1^{(1)} + p_{12}x_2^{(1)} \qquad \text{and} \qquad \frac{dx_2^{(1)}}{dt} = p_{21}x_1^{(1)} + p_{22}x_2^{(1)}.$$

Substituting the first equation above and its counterpart for $x^{(2)}$ into the first determinant appearing in dW/dt and evaluating the result yields

$$p_{11}\begin{vmatrix} x_1^{(1)} & x_1^{(2)} \\ x_2^{(1)} & x_2^{(2)} \end{vmatrix} = p_{11}W.$$

Similarly, the second determinant in dW/dt is evaluated as $p_{22}W$, yielding the desired result.

(c) From part (b) we have $dW/W = (p_{11}(t) + p_{22}(t))dt$, so $W(t) = ce^{\int (p_{11}(t)+p_{22}(t))\,dt}$.

(d) Follow the steps in parts (a), (b), and (c).

6.(a)

$$W = \begin{vmatrix} t & t^2 \\ 1 & 2t \end{vmatrix} = 2t^2 - t^2 = t^2.$$

(b) Pick $t = t_0$, then $c_1\mathbf{x}^{(1)}(t_0) + c_2\mathbf{x}^{(2)}(t_0) = \mathbf{0}$ implies $c_1t_0 + c_2t_0^2 = 0$ and $c_1 + 2c_2t_0 = 0$, which has a no-zero solution for c_1 and c_2 if and only if $t_0 \cdot 2t_0 - 1 \cdot t_0^2 = t_0^2 = 0$. Thus $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ are linearly independent at each point except at $t = 0$. Thus they are linearly independent on every interval.

(c) From part (a) we see that the Wronskian vanishes at $t = 0$, but not at any other point. By Theorem 7.4.3, if $\mathbf{P}(t)$, from Eq.(3) is continuous, then the Wronskian is either identically zero or else never vanishes. Hence we conclude that the differential equation satisfied by $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ must have at least one discontinuous coefficient at $t = 0$.

(d) To obtain the system satisfied by $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ we consider $\mathbf{x} = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t)$, or $x_1 = c_1 t + c_2 t^2$ and $x_2 = c_1 + c_2 2t$. Taking the derivatives of these we obtain $x_1' = c_1 + 2c_2 t$ and $x_2' = 2c_2$. Solving these for c_1 and c_2 we get $c_1 = x_1' - tx_2'$ and $c_2 = x_2'/2$. Thus $x_1 = tx_1' - (t^2/2)x_2'$ and $x_2 = x_1'$. Writing this system in matrix form we have

$$\mathbf{x} = \begin{pmatrix} t & -t^2/2 \\ 1 & 0 \end{pmatrix} \mathbf{x}'.$$

Finding the inverse of the matrix multiplying $\mathbf{x}'$ yields the desired solution. Note that at $t = 0$ two of the elements of $\mathbf{P}(t)$ are discontinuous.

7.5

1. Assuming that there are solutions of the form $\mathbf{x}= \boldsymbol{\xi}\, e^{rt}$, we substitute into the differential equation to find

$$r\boldsymbol{\xi}\, e^{rt} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \boldsymbol{\xi}\, e^{rt}.$$

Since

$$\boldsymbol{\xi} = \mathbf{I}\boldsymbol{\xi} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \boldsymbol{\xi},$$

we can write this equation (after dividing by the nonzero expression e^{rt}) as

$$\begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \boldsymbol{\xi} - r \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \boldsymbol{\xi} = \mathbf{0}$$

and thus we must solve the algebraic equations

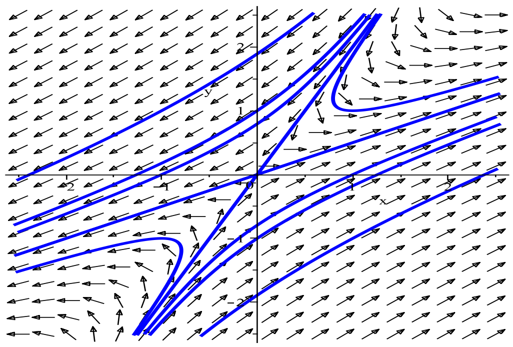
$$\begin{pmatrix} 3-r & -2 \\ 2 & -2-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

for r, ξ_1 and ξ_2. (Subsequent problems will be solved by the same method.) For a nonzero solution, we must have $\det(\mathbf{A} - r\mathbf{I}) = r^2 - r - 2 = 0$. The roots of the characteristic equation are the eigenvalues of the matrix: $r_1 = -1$ and $r_2 = 2$. For $r = -1$, the two equations reduce to $2\xi_1 = \xi_2$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (1, 2)^T$. Substitution of $r = 2$ results in the single equation $\xi_1 = 2\xi_2$. A corresponding eigenvector is $\boldsymbol{\xi}^{(2)} = (2, 1)^T$. Since the eigenvalues are distinct, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t}.$$

If the initial condition is a multiple of $(1, 2)^T$, then the solution will tend to the origin along the eigenvector $(1, 2)^T$. For $c_2 \neq 0$ all other solutions will tend to infinity asymptotic to the eigenvector $(2, 1)^T$.

(b)



5.(a) Setting $\mathbf{x} = \boldsymbol{\xi} e^{rt}$, and substituting into the ODE, we obtain the algebraic equations

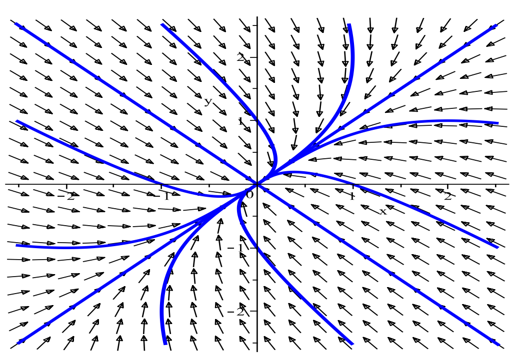
$$\begin{pmatrix} -2-r & 1 \\ 1 & -2-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $\det(\mathbf{A} - r\mathbf{I}) = r^2 + 4r + 3 = 0$. The roots of the characteristic equation are $r_1 = -1$ and $r_2 = -3$. For $r = -1$, the two equations reduce to $\xi_1 = \xi_2$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (1, 1)^T$. Substitution of $r = -3$ results in the single equation $\xi_1 + \xi_2 = 0$. A corresponding eigenvector is $\boldsymbol{\xi}^{(2)} = (1, -1)^T$. Since the eigenvalues are distinct, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t}.$$

If the initial condition is a multiple of $(1, -1)^T$, then the solution will tend to the origin along the eigenvector $(1, -1)^T$. Since e^{-t} is the dominant term as $t \to \infty$, as long as $c_1 \neq 0$, all trajectories approach the origin asymptotic to the eigenvector $(1, 1)^T$.

(b)



6.(a) Setting $\mathbf{x} = \boldsymbol{\xi}\, e^{rt}$, and substituting into the ODE, we obtain the algebraic equations

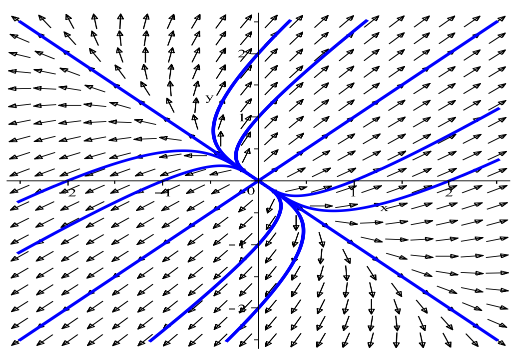
$$\begin{pmatrix} 5/4 - r & 3/4 \\ 3/4 & 5/4 - r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $\det(\mathbf{A} - r\mathbf{I}) = r^2 + (5/2)r + 1 = 0$. The roots of the characteristic equation are $r_1 = 2$ and $r_2 = 1/2$. For $r = 2$, the two equations reduce to $\xi_1 = \xi_2$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (1, 1)^T$. Substitution of $r = 1/2$ results in the single equation $\xi_1 + \xi_2 = 0$. A corresponding eigenvector is $\boldsymbol{\xi}^{(2)} = (1, -1)^T$. Since the eigenvalues are distinct, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t/2}.$$

The behavior of the solutions is similar to Problem 5, except the trajectories are reversed, since the roots are positive.

(b)



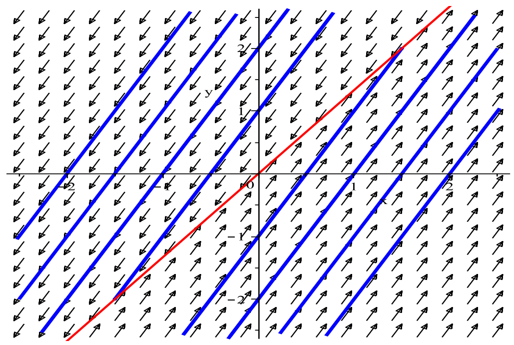
7.(a) Setting $\mathbf{x} = \boldsymbol{\xi}\, e^{rt}$ results in the algebraic equations

$$\begin{pmatrix} 4 - r & -3 \\ 8 & -6 - r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $\det(\mathbf{A} - r\mathbf{I}) = r^2 + 2r = 0$. The roots of the characteristic equation are $r_1 = -2$ and $r_2 = 0$. With $r = -2$, the system of equations reduces to $6\xi_1 - 3\xi_2 = 0$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (1, 2)^T$. For the case $r = 0$, the system is equivalent to the equation $4\xi_1 - 3\xi_2 = 0$. An eigenvector is $\boldsymbol{\xi}^{(2)} = (3, 4)^T$. Since the eigenvalues are distinct, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

(b)



The entire line along the eigendirection $\boldsymbol{\xi}^{(2)} = (3, 4)^T$ consists of constant solutions. All other solutions converge. The direction field changes across the line $4x_1 - 3x_2 = 0$. Eliminating the exponential terms in the solution, the trajectories are given by $2x_1 - x_2 = 2c_2$.

9. The characteristic equation is given by

$$\begin{vmatrix} 1-r & i \\ -i & 1-r \end{vmatrix} = (1-r)^2 + i^2 = r(r-2) = 0\,.$$

The equation has roots $r_1 = 0$ and $r_2 = 2$. For $r = 0$, the components of the solution vector must satisfy $\xi_1 + i\xi_2 = 0$. Thus the corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (1\,, i)^T$. Substitution of $r = 2$ results in the single equation $-\xi_1 + i\xi_2 = 0$. A corresponding eigenvector is $\boldsymbol{\xi}^{(2)} = (1\,, -i)^T$. Since the eigenvalues are distinct, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ i \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{2t}.$$

14. Setting $\mathbf{x}= \boldsymbol{\xi}\, e^{rt}$ results in the algebraic equations

$$\begin{pmatrix} 1-r & -1 & 4 \\ 3 & 2-r & -1 \\ 2 & 1 & -1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $\det(\mathbf{A} - r\mathbf{I}) = r^3 - 2r^2 - 5r + 6 = 0$. The roots of the characteristic equation are $r_1 = 1$, $r_2 = -2$ and $r_3 = 3$. Setting $r = 1$, we have

$$\begin{pmatrix} 0 & -1 & 4 \\ 3 & 1 & -1 \\ 2 & 1 & -2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This system is reduced to the equations $\xi_1 + \xi_3 = 0$ and $\xi_2 - 4\xi_3 = 0$. A corresponding solution vector is given by $\boldsymbol{\xi}^{(1)} = (1\,, -4\,, -1)^T$. In a similar way the eigenvectors corresponding to r_2 and r_3 are found to be $\boldsymbol{\xi}^{(2)} = (1\,, -1\,, -1)^T$ and

$\boldsymbol{\xi}^{(3)} = (1\,,2\,,1)^T$, respectively. Since the eigenvalues are distinct, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ -4 \\ -1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} e^{-2t} + c_3 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} e^{3t}.$$

16. Setting $\mathbf{x}= \boldsymbol{\xi}\, e^{rt}$ results in the algebraic equations

$$\begin{pmatrix} -2-r & 1 \\ -5 & 4-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $\det(\mathbf{A} - r\mathbf{I}) = r^2 - 2r - 3 = 0$. The roots of the characteristic equation are $r_1 = -1$ and $r_2 = 3$. With $r = -1$, the system of equations reduces to $\xi_1 - \xi_2 = 0$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (1\,,1)^T$. For the case $r = 3$, the system is equivalent to the equation $5\,\xi_1 - \xi_2 = 0$. An eigenvector is $\boldsymbol{\xi}^{(2)} = (1\,,5)^T$. Since the eigenvalues are distinct, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 5 \end{pmatrix} e^{3t}.$$

Invoking the initial conditions, we obtain the system of equations $c_1 + c_2 = 1$ and $c_1 + 5\,c_2 = 3$. Hence $c_1 = 1/2$ and $c_2 = 1/2$, and the solution of the IVP is

$$\mathbf{x} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + \frac{1}{2} \begin{pmatrix} 1 \\ 5 \end{pmatrix} e^{3t}.$$

As $t \to \infty$, the solution becomes asymptotic to $x_2 = 5x_1$.

20. Setting $\mathbf{x}= \boldsymbol{\xi}\, t^r$, for $t \neq 0$ results in the algebraic equations

$$\begin{pmatrix} 2-r & -1 \\ 3 & -2-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $\det(\mathbf{A} - r\mathbf{I}) = r^2 - 1 = 0$. The roots of the characteristic equation are $r_1 = 1$ and $r_2 = -1$. With $r = 1$, the system of equations reduces to $\xi_1 - \xi_2 = 0$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (1\,,1)^T$. For the case $r = -1$, the system is equivalent to the equation $3\,\xi_1 - \xi_2 = 0$. An eigenvector is $\boldsymbol{\xi}^{(2)} = (1\,,3)^T$. It follows that

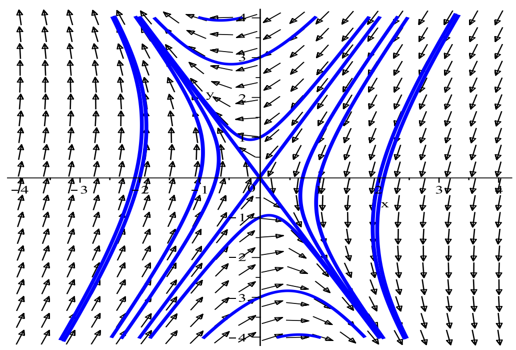
$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} t \text{ and } \mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} t^{-1}.$$

The Wronskian of this solution set is $W\left[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\right] = 2$. Thus the solutions are linearly independent for $t > 0$. Hence the general solution is

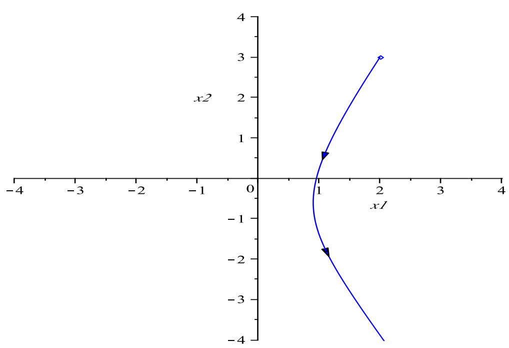
$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} t + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} t^{-1}.$$

25. (a) The general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-2t}.$$



(b)



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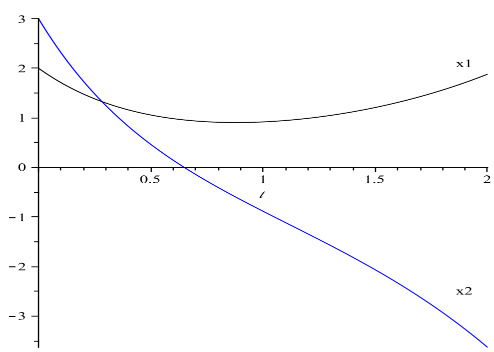
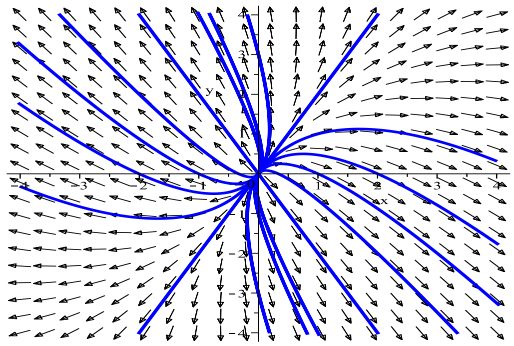
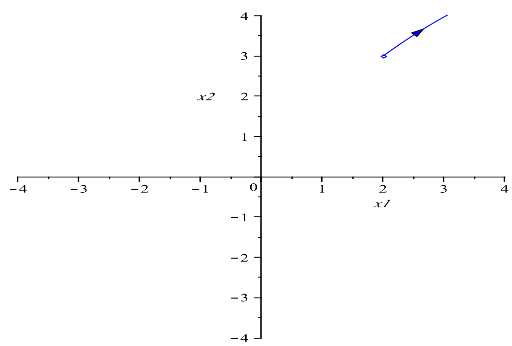


27.(a) The general solution is

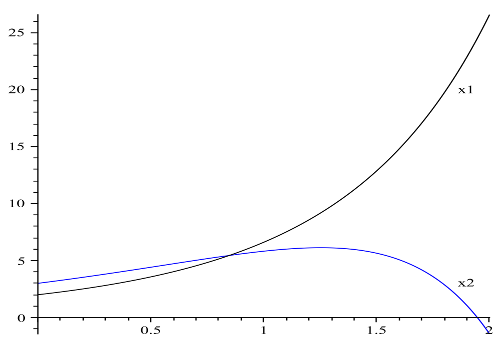
$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{2t}.$$



(b)



(c)



31.(a) For $\alpha = 1/2$, solution of the ODE requires that

$$\begin{pmatrix} -1-r & -1 \\ -1/2 & -1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation is $2r^2 + 4r + 1 = 0$, with roots $r_1 = -1 + 1/\sqrt{2}$ and $r_2 = -1 - 1/\sqrt{2}$. With $r = -1 + 1/\sqrt{2}$, the system of equations is $\sqrt{2}\xi_1 + 2\xi_2 = 0$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (-\sqrt{2}, 1)^T$. Substitution of $r = -1 - 1/\sqrt{2}$ results in the equation $\sqrt{2}\xi_1 - 2\xi_2 = 0$. An eigenvector is $\boldsymbol{\xi}^{(2)} = (\sqrt{2}, 1)^T$. The general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} e^{(-2+\sqrt{2})t/2} + c_2 \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix} e^{(-2-\sqrt{2})t/2}.$$

The eigenvalues are distinct and both negative. The equilibrium point is a stable node.

(b) For $\alpha = 2$, the characteristic equation is given by $r^2 + 2\,r - 1 = 0$, with roots $r_1 = -1 + \sqrt{2}$ and $r_2 = -1 - \sqrt{2}$. With $r = -1 + \sqrt{2}$, the system of equations reduces to $\sqrt{2}\,\xi_1 + \xi_2 = 0$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (1\,, -\sqrt{2})^T$. Substitution of $r = -1 - \sqrt{2}$ results in the equation $\sqrt{2}\,\xi_1 - \xi_2 = 0$. An eigenvector is $\boldsymbol{\xi}^{(2)} = (1\,, \sqrt{2})^T$. The general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} e^{(-1+\sqrt{2})t} + c_2 \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{(-1-\sqrt{2})t}.$$

The eigenvalues are of opposite sign, hence the equilibrium point is a saddle point.

(c) The eigenvalues are given by

$$\begin{vmatrix} -1-r & -1 \\ -\alpha & -1-r \end{vmatrix} = r^2 + 2r + 1 - \alpha = 0.$$

This $r_{1,2} = -1 \pm \sqrt{\alpha}$. Note that in part (a) the eigenvalues are both negative while in part (b) they differ in sign. Thus, if we choose $\alpha = 1$, then one eigenvalue is zero, which is the transition of the one root from negative to positive. This is the desired bifurcation point.

7.6

1. (a) Setting $\mathbf{x}= \boldsymbol{\xi}\, e^{rt}$ results in the algebraic equations

$$\begin{pmatrix} 3-r & -2 \\ 4 & -1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we require that $\det(\mathbf{A} - r\mathbf{I}) = r^2 - 2r + 5 = 0$. The roots of the characteristic equation are $r = 1 \pm 2i$. Substituting $r = 1 - 2i$, the two equations reduce to $(2+2i)\,\xi_1 - 2\,\xi_2 = 0$. The eigenvector is $\boldsymbol{\xi}^{(1)} = (1\,, 1+i)^T$. Hence one of the complex-valued solutions is given by

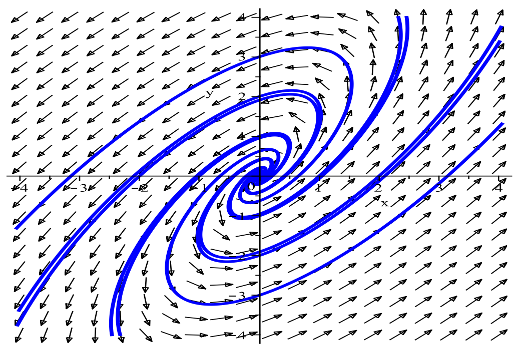
$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1+i \end{pmatrix} e^{(1-2i)t} = \begin{pmatrix} 1 \\ 1+i \end{pmatrix} e^t(\cos 2t - i\,\sin 2t) =$$

$$= e^t \begin{pmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{pmatrix} + i\, e^t \begin{pmatrix} -\sin 2t \\ -\sin 2t + \cos 2t \end{pmatrix}.$$

To find real-valued solutions (see Eqs.(10) and (11)) we take the real and imaginary parts, respectively, of $\mathbf{x}^{(1)}(t)$:

$$\mathbf{x} = c_1\, e^t \begin{pmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{pmatrix} + c_2\, e^t \begin{pmatrix} -\sin 2t \\ -\sin 2t + \cos 2t \end{pmatrix}.$$

(b)



The solutions spiral to ∞ as $t \to \infty$ due to the e^t terms.

7. The eigensystem is obtained from analysis of the equation

$$\begin{pmatrix} 1-r & 0 & 0 \\ 2 & 1-r & -2 \\ 3 & 2 & 1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The characteristic equation of the coefficient matrix is $(1-r)(r^2 - 2r + 5) = 0$, with roots $r_1 = 1$, $r_2 = 1 + 2i$ and $r_3 = 1 - 2i$. Setting $r = 1$, the equations reduce to $\xi_1 - \xi_3 = 0$ and $3\xi_1 + 2\xi_2 = 0$. If we choose $\xi_2 = -3$, the corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (2, -3, 2)^T$. With $r = 1 + 2i$, the system of equations is equivalent to $i\xi_1 = 0$ and $i\xi_2 + \xi_3 = 0$. An eigenvector is given by $\boldsymbol{\xi}^{(2)} = (0, 1, -i)^T$. Hence one of the complex-valued solutions is given by

$$\mathbf{x}^{(2)} = \begin{pmatrix} 0 \\ 1 \\ -i \end{pmatrix} e^{(1+2i)t} = \begin{pmatrix} 0 \\ 1 \\ -i \end{pmatrix} e^t(\cos 2t + i \sin 2t).$$

Taking the real and imaginary parts, we obtain

$$e^t \begin{pmatrix} 0 \\ \cos 2t \\ \sin 2t \end{pmatrix} \qquad \text{and} \qquad e^t \begin{pmatrix} 0 \\ \sin 2t \\ -\cos 2t \end{pmatrix}.$$

Thus the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} e^t + c_2\, e^t \begin{pmatrix} 0 \\ \cos 2t \\ \sin 2t \end{pmatrix} + c_3 e^t \begin{pmatrix} 0 \\ \sin 2t \\ -\cos 2t \end{pmatrix},$$

which spirals to ∞ about the x_1 axis in the $x_1x_2x_3$ space as $t \to \infty$ (for most initial conditions).

9. Solution of the system of ODEs requires that

$$\begin{pmatrix} 1-r & -5 \\ 1 & -3-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation is $r^2 + 2r + 2 = 0$, with roots $r = -1 \pm i$. Substituting $r = -1 + i$, the equations are equivalent to $\xi_1 = (2+i)\xi_2$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (2+i, 1)^T$. One of the complex-valued solutions is given by

$$\mathbf{x}^{(1)} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix} e^{(-1+i)t} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix} e^{-t}(\cos t + i\,\sin t) =$$

$$= e^{-t}\begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + ie^{-t}\begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix}.$$

Hence the general solution is

$$\mathbf{x} = c_1\,e^{-t}\begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + c_2\,e^{-t}\begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix}.$$

Invoking the initial conditions, we obtain the system of equations $2c_1 + c_2 = 1$ and $c_1 = 1$. Solving for the coefficients, the solution of the initial value problem is

$$\mathbf{x} = e^{-t}\begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} - e^{-t}\begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix} = e^{-t}\begin{pmatrix} \cos t - 3\sin t \\ \cos t - \sin t \end{pmatrix},$$

which spirals to zero as $t \to \infty$, due to the e^{-t} term.

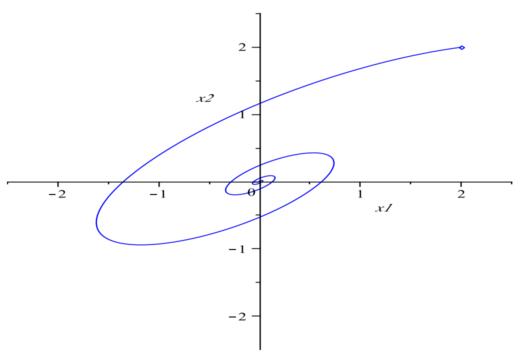
11.(a) The eigenvalues are given by

$$\begin{vmatrix} 3/4 - r & -2 \\ 1 & -5/4 - r \end{vmatrix} = r^2 + r/2 + 17/16 = 0,$$

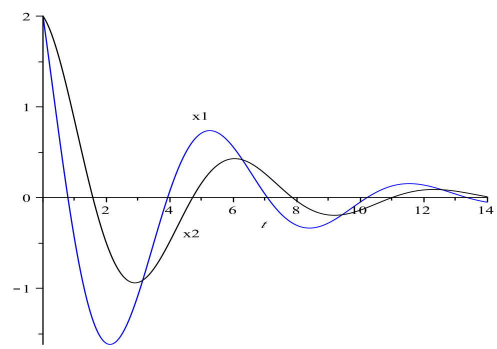
so $r = 1/4 \pm i$. With $\mathbf{x}(0) = (2\ , 2)^T$, the solution is

$$\mathbf{x} = e^{-t/4}\begin{pmatrix} 2\ \cos\ t - 2\ \sin\ t \\ 2\ \cos\ t \end{pmatrix}.$$

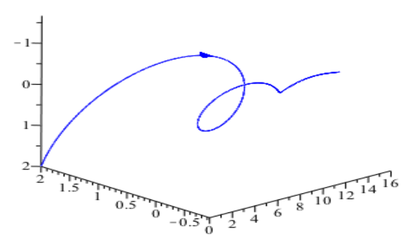
(b)



(c)



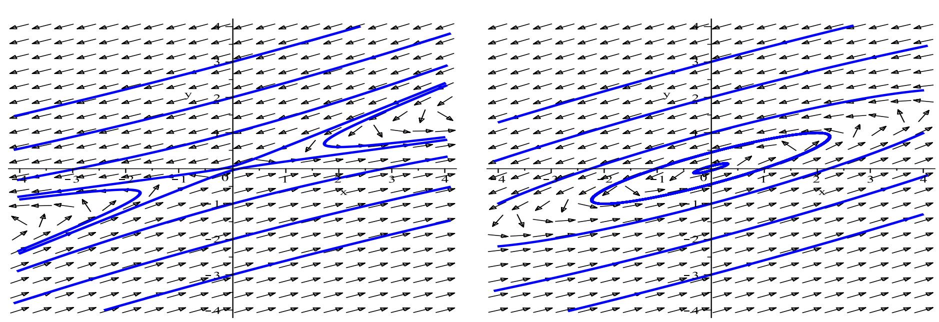
(d) Choose $\mathbf{x}(0) = (2\ ,2)^T$, then the trajectory starts at $(2, 2)$ in the x_1x_2 plane and spirals around the t-axis and converges to the t-axis as $t \to \infty$.



15.(a) The roots of the characteristic equation, $r^2 - 4 + 5\alpha = 0$, are $r_{1,2} = \pm\sqrt{4 - 5\alpha}$.

(b) The qualitative nature of the phase portrait changes when r goes from real to complex. Thus $\alpha = 4/5$ is the critical value.

(c)

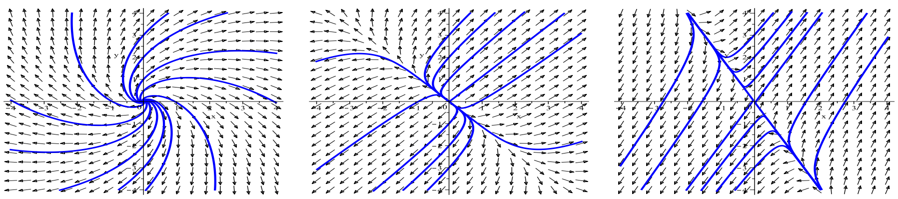


(a) $\alpha = 3/5$ (b) $\alpha = 1$

16.(a) The roots of the characteristic equation, $r^2 - 5r/2 + (25/16 - 3\alpha/4) = 0$, are $r_{1,2} = 5/4 \pm \sqrt{3\alpha}/2$.

(b) There are two critical values of α. For $\alpha < 0$ the eigenvalues are complex, while for $\alpha > 0$ they are real. There will be a second critical value of α when $r_2 = 0$, or $\alpha = 25/12$. In this case the second real eigenvalue goes from positive to negative.

(c)

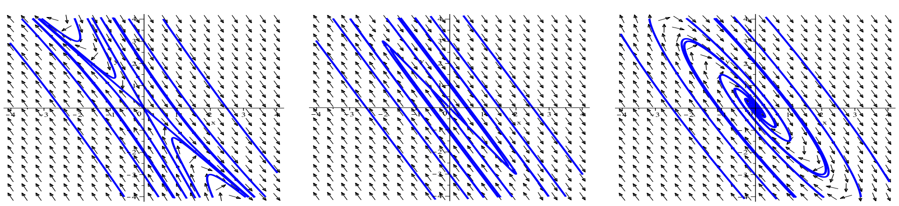


(a) $\alpha = -1$ (b) $\alpha = 1$ (c) $\alpha = 3$

18.(a) The roots of the characteristic equation, $r^2 + r - 12 + 6\alpha = 0$, are $r_{1,2} = -1/2 \pm \sqrt{49 - 24\alpha}/2$.

(b) There are two critical values of α. The first occurs when $49 - 24\alpha = 1$ (in which case $r_2 = 0$) and when $49 - 24\alpha = 0$, in which case $r_1 = r_2 = -1/2$. Thus the critical values are at $\alpha = 2$ and $\alpha = 49/24$.

(c)



(a) $\alpha = 1.8$ (b) $\alpha = 2.01$ (c) $\alpha = 2.1$

21. Based on the method in Problem 19 of Section 7.5, setting $\mathbf{x} = \boldsymbol{\xi}\, t^r$ and assuming $t \neq 0$ results in the algebraic equations

$$\begin{pmatrix} -1 - r & -1 \\ 2 & -1 - r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation for the system is $r^2+2r+3=0$, with roots $r_{1,2}=-1\pm\sqrt{2}i$. With $r=-1+\sqrt{2}i$, the equations reduce to the single equation $\sqrt{2}i\xi_1+\xi_2=0$. A corresponding eigenvector is $\boldsymbol{\xi}^{(1)}=(1\,,-\sqrt{2}i)^T$. One complex-valued solution is

$$\mathbf{x}^{(1)}=\begin{pmatrix}1\\-\sqrt{2}i\end{pmatrix}t^{-1+\sqrt{2}i}\,.$$

We can write $t^{\sqrt{2}i}=e^{\sqrt{2}i\ln t}$. Hence

$$\mathbf{x}^{(1)}=\begin{pmatrix}1\\-\sqrt{2}i\end{pmatrix}t^{-1}e^{\sqrt{2}i\;\ln t}=\begin{pmatrix}1\\-\sqrt{2}i\end{pmatrix}t^{-1}\left[\cos(\sqrt{2}\ln t)+i\,\sin(\sqrt{2}\ln t)\right].$$

Separating the complex valued solution into real and imaginary parts, we obtain that the general solution is

$$\mathbf{x}=c_1t^{-1}\begin{pmatrix}\cos(\sqrt{2}\ln t)\\\sqrt{2}\sin(\sqrt{2}\ln t)\end{pmatrix}+c_2t^{-1}\begin{pmatrix}\sin(\sqrt{2}\ln t)\\-\sqrt{2}\cos(\sqrt{2}\ln t)\end{pmatrix}.$$

23.(a) The characteristic equation of the system is $(r+1/4)((r+1/4)^2+1)=0$, with eigenvalues $r_1=-1/4$, and $r_{2,3}=-1/4\pm i$. For $r=-1/4$, simple calculations reveal that a corresponding eigenvector is $\boldsymbol{\xi}^{(1)}=(0,0,1)^T$. Setting $r=-1/4-i$, we obtain the system of equations $\xi_1-i\,\xi_2=0$ and $\xi_3=0$. A corresponding eigenvector is $\boldsymbol{\xi}^{(2)}=(i\,,1\,,0)^T$. Hence one solution is

$$\mathbf{x}^{(1)}=\begin{pmatrix}0\\0\\1\end{pmatrix}e^{-t/4}.$$

Another solution, which is complex-valued, is given by

$$\mathbf{x}^{(2)}=\begin{pmatrix}i\\1\\0\end{pmatrix}e^{-(\frac{1}{4}-i)t}=\begin{pmatrix}i\\1\\0\end{pmatrix}e^{-t/4}(\cos\,t-i\,\sin t)=$$

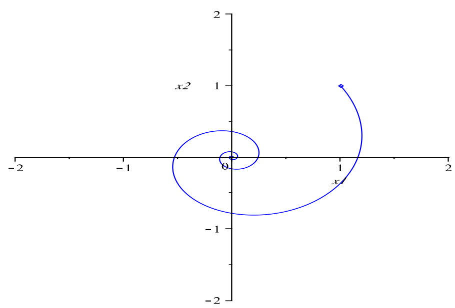
$$=e^{-t/4}\begin{pmatrix}\sin\,t\\\cos\,t\\0\end{pmatrix}+ie^{-t/4}\begin{pmatrix}\cos\,t\\-\sin\,t\\0\end{pmatrix}.$$

Using the real and imaginary parts of $\mathbf{x}^{(2)}$, the general solution is constructed as

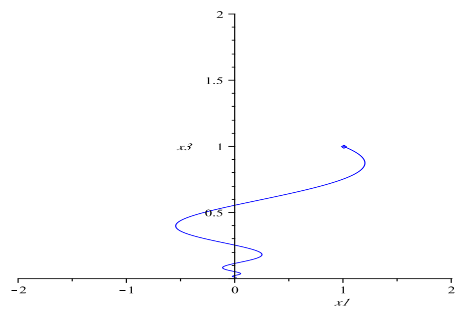
$$\mathbf{x}=c_1\begin{pmatrix}0\\0\\1\end{pmatrix}e^{-t/4}+c_2\,e^{-t/4}\begin{pmatrix}\sin\,t\\\cos\,t\\0\end{pmatrix}+c_3\,e^{-t/4}\begin{pmatrix}\cos\,t\\-\sin\,t\\0\end{pmatrix}.$$

(b) With $\mathbf{x}(0)=(1\,,1\,,1)$, the solution of the initial value problem is

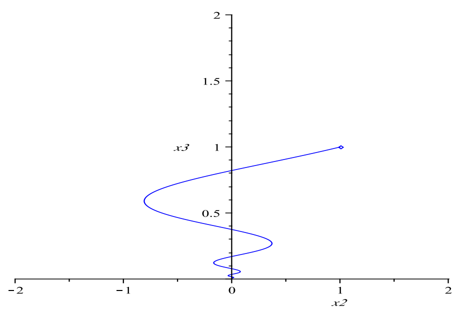
$$\mathbf{x}=\begin{pmatrix}0\\0\\e^{-t/4}\end{pmatrix}+e^{-t/4}\begin{pmatrix}\sin\,t+\cos\,t\\\cos\,t-\sin\,t\\0\end{pmatrix}.$$



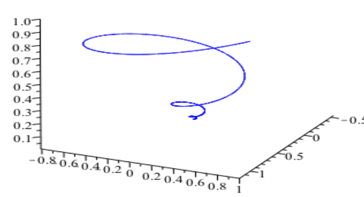
(a) $x_1 - x_2$



(b) $x_1 - x_3$



(c) $x_2 - x_3$



(d) $x_1x_2x_3$

30.(a) Following the steps leading to Eq.(24) and using the given values for the m's and k's, we have

$$\begin{aligned}
y_1' &= y_3 \\
y_2' &= y_4 \\
y_3' &= -4y_1 + 3y_2 \\
y_4' &= (9/4)y_1 - (13/4)y_2
\end{aligned}$$

hence the coefficient matrix is

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 3 & 0 & 0 \\ 9/4 & -13/4 & 0 & 0 \end{pmatrix}.$$

(b) The eigenvalues and corresponding eigenvectors of $\mathbf{A}$ are:

$$\begin{aligned}
&r_1 = i\,, \quad \boldsymbol{\xi}^{(1)} = (1, 1, i, i)^T \\
&r_2 = -i\,, \quad \boldsymbol{\xi}^{(2)} = (1, 1, -i, -i)^T \\
&r_3 = (5/2)\,i\,, \quad \boldsymbol{\xi}^{(3)} = (4, -3, 10i, -15i/2)^T \\
&r_4 = -(5/2)\,i\,, \quad \boldsymbol{\xi}^{(4)} = (4, -3, -10i, 15i/2)^T
\end{aligned}$$

(c) Note that

$$\boldsymbol{\xi}^{(1)}e^{it} = \begin{pmatrix} 1 \\ 1 \\ i \\ i \end{pmatrix} (\cos\, t + i\, \sin\, t)$$

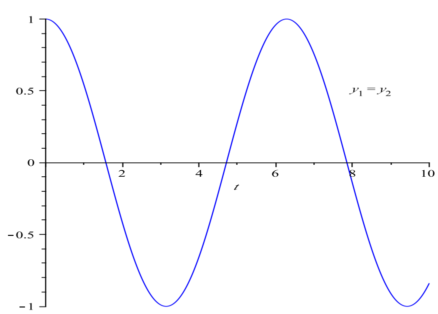
and

$$\boldsymbol{\xi}^{(3)}e^{(5/2)it} = \begin{pmatrix} 4 \\ -3 \\ 10i \\ -15i/2 \end{pmatrix} (\cos 5t/2 + i\, \sin 5t/2).$$

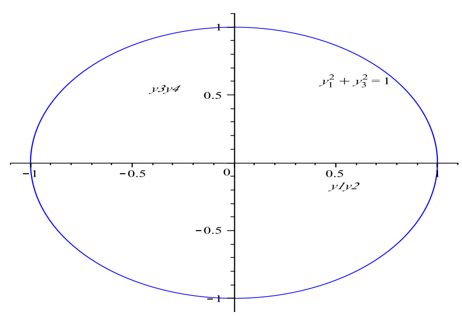
Hence the general solution is

$$\mathbf{y} = c_1 \begin{pmatrix} \cos\, t \\ \cos\, t \\ -\sin\, t \\ -\sin\, t \end{pmatrix} + c_2 \begin{pmatrix} \sin\, t \\ \sin\, t \\ \cos\, t \\ \cos\, t \end{pmatrix} + c_3 \begin{pmatrix} 4\cos 5t/2 \\ -3\cos 5t/2 \\ -10\sin 5t/2 \\ (15/2)\sin 5t/2 \end{pmatrix} + c_4 \begin{pmatrix} 4\sin 5t/2 \\ -3\sin 5t/2 \\ 10\cos 5t/2 \\ -(15/2)\cos 5t/2 \end{pmatrix}.$$

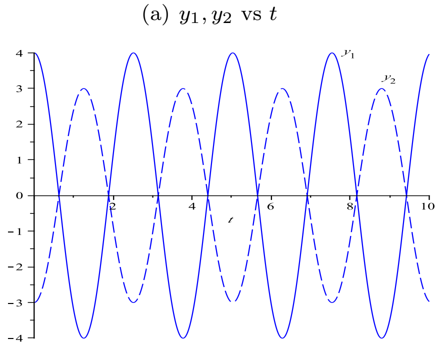
(d) The two modes have natural frequencies of $\omega_1 = 1$ rad/sec and $\omega_2 = 5/2$ rad/sec. The first pair of figures corresponds to $c_1 = 1$, $c_2 = c_3 = c_4 = 0$ (first mode - observe that in this case $y_1 = y_2$ and $y_3 = y_4$), the second pair corresponds to $c_3 = 1$, $c_1 = c_2 = c_4 = 0$ (second mode).



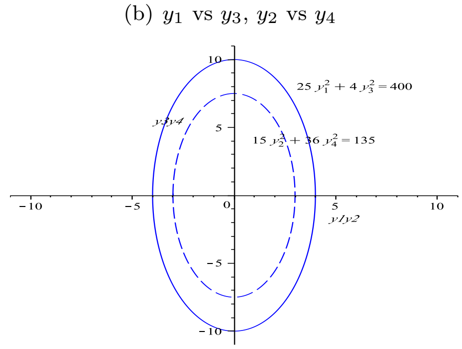
(a) y_1, y_2 vs t



(b) y_1 vs y_3, y_2 vs y_4



(c) y_1, y_2 vs t

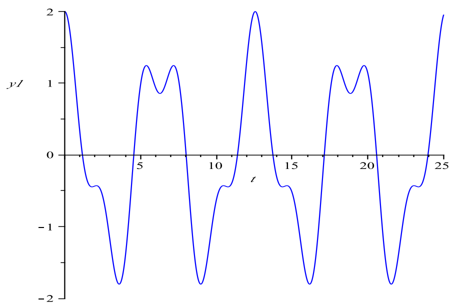


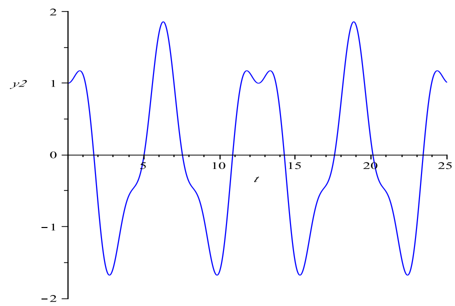
(d) y_1 vs y_3, y_2 vs y_4

(e) For the initial condition $\mathbf{y}(0) = (2,1,0,0)^T$, it is necessary that

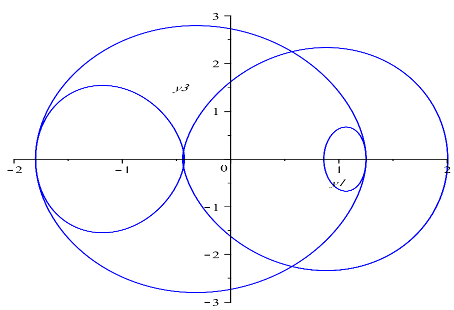
$$\begin{pmatrix}2\\1\\0\\0\end{pmatrix} = c_1\begin{pmatrix}1\\1\\0\\0\end{pmatrix} + c_2\begin{pmatrix}0\\0\\1\\1\end{pmatrix} + c_3\begin{pmatrix}4\\-3\\0\\0\end{pmatrix} + c_4\begin{pmatrix}0\\0\\10\\-15/2\end{pmatrix},$$

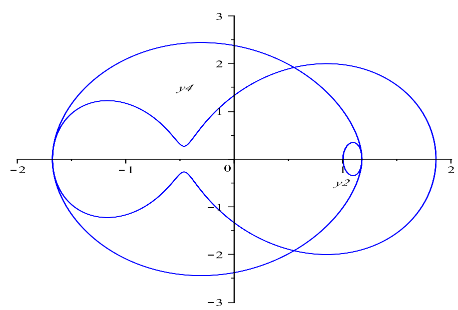
resulting in the coefficients $c_1 = 10/7$, $c_2 = 0$, $c_3 = 1/7$ and $c_4 = 0$.





The solutions are periodic with period 4π.





7.7

Each of the problems 1 through 10, except 2 and 8, has been solved in one of the previous sections. Thus a fundamental matrix for the given systems can be readily written down. The fundamental matrix $\mathbf{\Phi}(t)$ satisfying $\mathbf{\Phi}(0) = \mathbf{I}$ can then be found as shown in the following problems.

2.(a) The characteristic equation is given by $r^2 + 3r/2 + 1/2 = 0$, so $r_1 = -1$, $r_2 = -1/2$. The corresponding eigenvectors are

$$r_1 = -1, \quad \boldsymbol{\xi}^{(1)} = \begin{pmatrix}-2\\1\end{pmatrix}; \quad r_2 = -1/2, \quad \boldsymbol{\xi}^{(2)} = \begin{pmatrix}2\\1\end{pmatrix}.$$

The general solution is

$$\mathbf{x} = c_1\begin{pmatrix}-2e^{-t}\\e^{-t}\end{pmatrix} + c_2\begin{pmatrix}2\,e^{-t/2}\\e^{-t/2}\end{pmatrix}.$$

Hence a fundamental matrix is given by

$$\boldsymbol{\Psi}(t) = \begin{pmatrix} -2e^{-t} & 2\,e^{-t/2} \\ e^{-t} & e^{-t/2} \end{pmatrix}.$$

(b) We now have

$$\boldsymbol{\Psi}(0) = \begin{pmatrix} -2 & 2 \\ 1 & 1 \end{pmatrix} \text{ and } \boldsymbol{\Psi}^{-1}(0) = \frac{1}{4}\begin{pmatrix} -1 & 2 \\ 1 & 2 \end{pmatrix},$$

so that

$$\boldsymbol{\Phi}(t) = \boldsymbol{\Psi}(t)\boldsymbol{\Psi}^{-1}(0) = \frac{1}{4}\begin{pmatrix} 2e^{-t} + 2e^{-t/2} & -4e^{-t} + 4e^{-t/2} \\ -e^{-t} + e^{-t/2} & 2e^{-t} + 2e^{-t/2} \end{pmatrix}.$$

4.(a) From Problem 4 of Section 7.5, we have the two linearly independent solutions

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} \quad \text{and} \quad \mathbf{x}^{(2)}(t) = \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t}.$$

Thus

$$\boldsymbol{\Psi}(t) = \begin{pmatrix} e^{2t} & e^{-3t} \\ e^{2t} & -4e^{-3t} \end{pmatrix}.$$

(b) We now have

$$\boldsymbol{\Psi}(0) = \begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix} \text{ and } \boldsymbol{\Psi}^{-1}(0) = \frac{1}{5}\begin{pmatrix} 4 & 1 \\ 1 & -1 \end{pmatrix},$$

so that

$$\boldsymbol{\Phi}(t) = \boldsymbol{\Psi}(t)\boldsymbol{\Psi}^{-1}(0) = \frac{1}{5}\begin{pmatrix} e^{-3t} + 4e^{2t} & -e^{-3t} + e^{2t} \\ -4e^{-3t} + 4e^{2t} & 4e^{-3t} + e^{2t} \end{pmatrix}.$$

6.(a) Two linearly independent real-valued solutions were found of the differential equation were found in Problem 2 of Section 7.6. Using the result of that problem, we have

$$\mathbf{x}^{(1)}(t) = e^{-t}\begin{pmatrix} -2\,\sin\,2t \\ \cos\,2t \end{pmatrix} \quad \text{and} \quad \mathbf{x}^{(2)}(t) = e^{-t}\begin{pmatrix} 2\,\cos\,2t \\ \sin\,2t \end{pmatrix}.$$

Thus

$$\boldsymbol{\Psi}(t) = \begin{pmatrix} -2e^{-t}\sin 2t & 2e^{-t}\cos 2t \\ e^{-t}\cos 2t & e^{-t}\sin 2t \end{pmatrix}.$$

(b) We now have

$$\boldsymbol{\Psi}(0) = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} \text{ and } \boldsymbol{\Psi}^{-1}(0) = \frac{1}{2}\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix},$$

so that

$$\boldsymbol{\Phi}(t) = \boldsymbol{\Psi}(t)\boldsymbol{\Psi}^{-1}(0) = \begin{pmatrix} e^{-t}\cos 2t & -2e^{-t}\sin 2t \\ e^{-t}\sin 2t/2 & e^{-t}\cos 2t \end{pmatrix}.$$

10.(a) From Problem 14 of Section 7.5 we have

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ -4 \\ -1 \end{pmatrix} e^{t}, \quad \mathbf{x}^{(2)}(t) = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} e^{-2t}, \quad \mathbf{x}^{(3)}(t) = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} e^{3t}$$

Thus

$$\mathbf{\Psi}(t) = \begin{pmatrix} e^{t} & e^{-2t} & e^{3t} \\ -4e^{t} & -e^{-2t} & 2e^{3t} \\ -e^{t} & -e^{-2t} & e^{3t} \end{pmatrix}.$$

(b) For the first column of $\mathbf{\Phi}$ we want to choose c_1, c_2 and c_3 such that $c_1\mathbf{x}^{(1)}(0) + c_2\mathbf{x}^{(2)}(0) + c_3\mathbf{x}^{(3)}(0) = (1, 0, 0)^T$. Thus $c_1 + c_2 + c_3 = 1$, $-4c_1 - c_2 + 2c_3 = 0$ and $-c_1 - c_2 + c_3 = 0$, which yield $c_1 = 1/6$, $c_2 = 1/3$ and $c_3 = 1/2$. The first column of $\mathbf{\Phi}$ is then

$$\begin{pmatrix} e^{t}/6 + e^{-2t}/3 + e^{3t}/2 \\ -2e^{t}/3 - e^{-2t}/3 + e^{3t} \\ -e^{t}/6 - e^{-2t}/3 + e^{3t}/2 \end{pmatrix}.$$

Likewise, for the second column we have $d_1\mathbf{x}^{(1)}(0) + d_2\mathbf{x}^{(2)}(0) + d_3\mathbf{x}^{(3)}(0) = (0, 1, 0)^T$, which yields $d_1 = -1/3$, $d_2 = 1/3$ and $d_3 = 0$. Finally, the coefficients for the third column of $\mathbf{\Phi}$ are given by $e_1 = 1/2$, $e_2 = -1$ and $e_3 = 1/2$. These give us

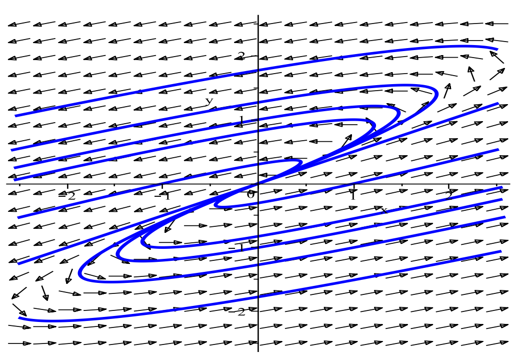
$$\mathbf{\Phi} = \begin{pmatrix} e^{t}/6 + e^{-2t}/3 + e^{3t}/2 & e^{-2t}/3 - e^{t}/3 & -e^{-2t} + e^{t}/2 + e^{3t}/2 \\ -2e^{t}/3 - e^{-2t}/3 + e^{3t} & 4e^{t}/3 - e^{-2t}/3 & -2e^{t} + e^{-2t} + e^{3t} \\ -e^{t}/6 - e^{-2t}/3 + e^{3t}/2 & e^{-t}/3 - e^{-2t}/3 & e^{-2t} - e^{-t}/2 + e^{3t}/2 \end{pmatrix}.$$

11. From Eq.(14) the solution is given by $\mathbf{\Phi}(t)\mathbf{x}^0$. Thus

$$\mathbf{x} = \frac{1}{2}\begin{pmatrix} 3e^{t} - e^{-t} & -e^{t} + e^{-t} \\ 3e^{t} - 3e^{-t} & -e^{t} + 3e^{-t} \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 7e^{t}/2 - 3e^{-t}/2 \\ 7e^{t}/2 - 9e^{-t}/2 \end{pmatrix}.$$

7.8

1.(a)



(b) From the general solution we have $x_2/x_1 = (c_1 + c_2 t)/(2c_1 + 2c_2 t + c_2)$, so that $\lim_{t\to\infty} x_2/x_1 = 1/2$. Thus all solutions (except the trivial one) diverge to infinity along lines of slope $1/2$ which can be seen in the trajectories shown in part (a).

(c) Solution of the ODE requires analysis of the algebraic equations

$$\begin{pmatrix} 3-r & -4 \\ 1 & -1-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $\det(\mathbf{A} - r\mathbf{I}) = r^2 - 2r + 1 = 0$. The only root is $r = 1$, which is an eigenvalue of multiplicity two. Setting $r = 1$ in the coefficient matrix reduces the system to the single equation $\xi_1 - 2\xi_2 = 0$. Hence the corresponding eigenvector is $\boldsymbol{\xi} = (2\,, 1)^T$. One solution is

$$\mathbf{x}^{(1)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^t.$$

In order to obtain a second linearly independent solution, we assume, as in Eq.(13), that $\mathbf{x} = \boldsymbol{\xi} t e^t + \boldsymbol{\eta} e^t$. As in Example 2, we find that $\boldsymbol{\xi}$ is an eigenvector, so we choose $\boldsymbol{\xi} = (2, 1)^T$. Then $\boldsymbol{\eta}$ must satisfy Eq.(24): $(\mathbf{A} - r\mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}$, or

$$\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

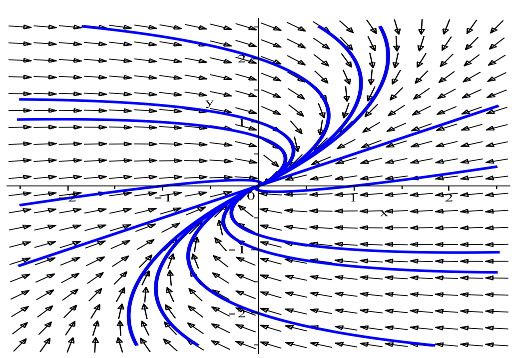
These equations reduce to $\eta_1 - 2\eta_2 = 1$. Set $\eta_2 = k$, some arbitrary constant. Then $\eta_1 = 1 + 2k$. A second solution is

$$\mathbf{x}^{(2)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} t e^t + \begin{pmatrix} 1+2k \\ k \end{pmatrix} e^t = \begin{pmatrix} 2 \\ 1 \end{pmatrix} t e^t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + k\begin{pmatrix} 2 \\ 1 \end{pmatrix} e^t.$$

Since the last term is a multiple of the first solution, it can be dropped. The general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^t + c_2 \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} t e^t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t \right].$$

3.(a)



(b) The origin is attracting. That is, as $t \to \infty$ the solution approaches the origin tangent to the line $x_2 = x_1/2$, which is obtained by taking the $\lim_{t\to\infty} x_2/x_1$ similar to Problem 1.

(c) Solution of the ODEs is based on the analysis of the algebraic equations

$$\begin{pmatrix} -\frac{3}{2} - r & 1 \\ -\frac{1}{4} & -\frac{1}{2} - r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation is $r^2 + 2\,r + 1 = 0$, with a single root $r = -1$. Setting $r = -1$, the two equations reduce to $\xi_1 - 2\xi_2 = 0$. The corresponding eigenvector is $\boldsymbol{\xi} = (2\,,1)^T$. One solution is

$$\mathbf{x}^{(1)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t}.$$

As in Problem 1, a second linearly independent solution is obtained by finding a generalized eigenvector. We therefore analyze the system

$$\begin{pmatrix} -1/2 & 1 \\ -1/4 & 1/2 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

The equations reduce to the single equation $-\eta_1 + 2\eta_2 = 4$. Let $\eta_1 = 2k$. We obtain $\eta_2 = 2 + k$, and a second linearly independent solution is

$$\mathbf{x}^{(2)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} te^{-t} + \begin{pmatrix} 2k \\ 2+k \end{pmatrix} e^{-t} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} te^{-t} + \begin{pmatrix} 0 \\ 2 \end{pmatrix} e^{-t} + k \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t}.$$

Dropping the last term, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t} + c_2 \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} te^{-t} + \begin{pmatrix} 0 \\ 2 \end{pmatrix} e^{-t} \right].$$

5. The eigensystem is obtained from analysis of the equation

$$\begin{pmatrix} 1-r & 1 & 1 \\ 2 & 1-r & -1 \\ 0 & -1 & 1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The characteristic equation of the coefficient matrix is $-r^3 + 3r^2 - 4 = 0$, with roots $r_1 = -1$ and $r_{2,3} = 2$. Setting $r = -1$, we have

$$\begin{pmatrix} 2 & 1 & 1 \\ 2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This system is reduced to the equations $2\xi_1 + \xi_2 + \xi_3 = 0$ and $\xi_2 - 2\xi_3 = 0$. A corresponding eigenvector is given by $\boldsymbol{\xi}^{(1)} = (-3\,,4\,,2)^T$. Setting $r = 2$, the system of equations is reduced to the equations $-\xi_1 + \xi_2 + \xi_3 = 0$ and $\xi_2 + \xi_3 = 0$. An eigenvector vector is given by $\boldsymbol{\xi}^{(2)} = (0\,,1\,,-1)^T$. The second solution corresponding to the double eigenvalue will have the form specified by Eq.(13), which yields

$$\mathbf{x}^{(3)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} te^{2t} + \boldsymbol{\eta} e^{2t}.$$

Substituting this into the given system, or using Eq.(24), we find that

$$\begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

Using row reduction we find that $\eta_1 = 1$ and $\eta_2 + \eta_3 = 1$. If we choose $\eta_2 = 0$, then $\boldsymbol{\eta} = (1, 0, 1)^T$ and thus

$$\mathbf{x}^{(3)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} te^{2t} + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{2t}.$$

Therefore the general solution may be written as

$$\mathbf{x} = c_1 \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{2t} + c_3 \left[\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} te^{2t} + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{2t} \right].$$

9. (a) Solution of the ODEs is based on the analysis of the algebraic equations

$$\begin{pmatrix} 2-r & \frac{3}{2} \\ -\frac{3}{2} & -1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation is $r^2 - r + 1/4 = 0$, with a single root $r = 1/2$. Setting $r = 1/2$, the two equations reduce to $\xi_1 + \xi_2 = 0$. The corresponding eigenvector is $\boldsymbol{\xi} = (1\,,-1)^T$. One solution is

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t/2}.$$

A second linearly independent solution is obtained by solving the system

$$\begin{pmatrix} 3/2 & 3/2 \\ -3/2 & -3/2 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The equations reduce to the single equation $3\eta_1 + 3\eta_2 = 2$. Let $\eta_1 = k$. We obtain $\eta_2 = 2/3 - k$, and a second linearly independent solution is

$$\mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} te^{t/2} + \begin{pmatrix} k \\ 2/3 - k \end{pmatrix} e^{t/2} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} te^{-t} + \begin{pmatrix} 0 \\ 2/3 \end{pmatrix} e^{t/2} + k \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t/2}.$$

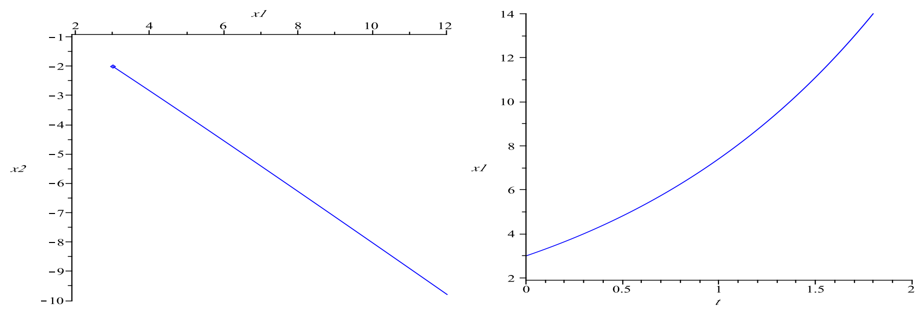
Dropping the last term, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t/2} + c_2 \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} te^{t/2} + \begin{pmatrix} 0 \\ 2/3 \end{pmatrix} e^{t/2} \right].$$

Imposing the initial conditions, we find that $c_1 = 3$, $-c_1 + 2c_2/3 = -2$, so that $c_1 = 3$ and $c_2 = 3/2$. Therefore the solution of the IVP is

$$\mathbf{x} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} e^{t/2} + \begin{pmatrix} 3/2 \\ -3/2 \end{pmatrix} te^{t/2}.$$

(b)



11.(a) The characteristic equation of the system is $(r-1)^2(r-2) = 0$. The eigenvalues are $r_1 = 2$ and $r_{2,3} = 1$. The eigenvector associated with r_1 is $\boldsymbol{\xi}^{(1)} = (0\,,0\,,1)^T$. Setting $r = 1$, the system of equations is reduced to the equations $\xi_1 = 0$ and $6\xi_2 + \xi_3 = 0$. An eigenvector vector is given by $\boldsymbol{\xi}^{(2)} = (0\,,1\,,-6)^T$. The second solution corresponding to the double eigenvalue will have the form specified by Eq.(13), which yields

$$\mathbf{x}^{(3)} = \begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix} te^t + \boldsymbol{\eta} e^t.$$

Substituting this into the given system, or using Eq.(24), we find that

$$\begin{pmatrix} 0 & 0 & 0 \\ -4 & 0 & 0 \\ 3 & 6 & 1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix}.$$

Using row reduction we find that $\eta_1 = -1/4$ and $6\eta_2 + \eta_3 = -21/4$. If we choose $\eta_2 = 0$, then $\boldsymbol{\eta} = (-1/4, 0, -21/4)^T$ and thus

$$\mathbf{x}^{(3)} = \begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix} te^t + \begin{pmatrix} -1/4 \\ 0 \\ -21/4 \end{pmatrix} e^t.$$

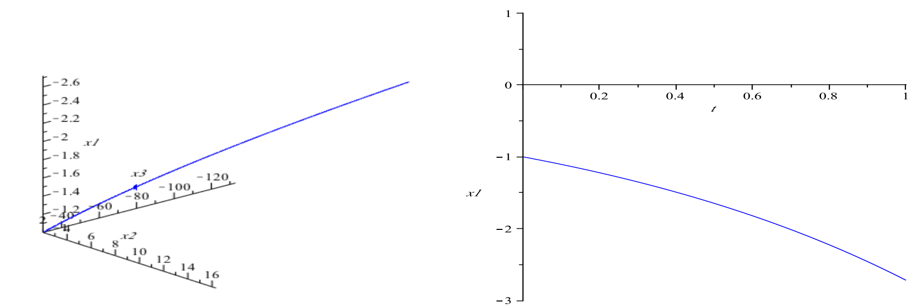
Therefore the general solution may be written as

$$\mathbf{x} = c_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix} e^t + c_3 \left[\begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix} te^t + \begin{pmatrix} -1/4 \\ 0 \\ -21/4 \end{pmatrix} e^t \right].$$

The initial conditions then yield $c_1 = 3$, $c_2 = 2$ and $c_3 = 4$ and hence

$$\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} e^{2t} + 4 \begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix} te^t + \begin{pmatrix} -1 \\ 2 \\ -33 \end{pmatrix} e^t.$$

(b)



12.(a) The characteristic equation of the system is $8\,r^3+60\,r^2+126\,r+49=0$. The eigenvalues are $r_1=-1/2$ and $r_{2,3}=-7/2$. The eigenvector associated with r_1 is $\boldsymbol{\xi}^{(1)}=(1\,,1\,,1)^T$. Setting $r=-7/2$, the components of the eigenvectors must satisfy the relation

$$\xi_1+\xi_2+\xi_3=0\,.$$

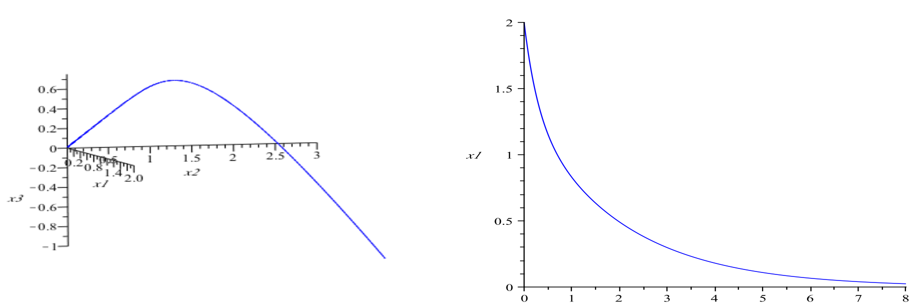
An eigenvector vector is given by $\boldsymbol{\xi}^{(2)}=(1\,,0\,,-1)^T$. Since the last equation has two free variables, a third linearly independent eigenvector (associated with $r=-7/2$) is $\boldsymbol{\xi}^{(3)}=(0\,,1\,,-1)^T$. Therefore the general solution may be written as

$$\mathbf{x}=c_1\begin{pmatrix}1\\1\\1\end{pmatrix}e^{-t/2}+c_2\begin{pmatrix}1\\0\\-1\end{pmatrix}e^{-7t/2}+c_3\begin{pmatrix}0\\1\\-1\end{pmatrix}e^{-7t/2}.$$

Invoking the initial conditions, we require that $c_1+c_2=2$, $c_1+c_3=3$, and $c_1-c_2-c_3=-1$. Hence the solution of the IVP is

$$\mathbf{x}=\frac{4}{3}\begin{pmatrix}1\\1\\1\end{pmatrix}e^{-t/2}+\frac{2}{3}\begin{pmatrix}1\\0\\-1\end{pmatrix}e^{-7t/2}+\frac{5}{3}\begin{pmatrix}0\\1\\-1\end{pmatrix}e^{-7t/2}.$$

(b)



14. Setting $\mathbf{x}=\boldsymbol{\xi}\,t^r$ and assuming $t\neq 0$ results in the algebraic equations

$$\begin{pmatrix}1-r & -4\\ 4 & -7-r\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}.$$

The characteristic equation is $r^2 + 6r + 9 = 0$, with a single root of $r_{1,2} = -3$. With $r = -3$, the system reduces to a single equation $\xi_1 - \xi_2 = 0$. An eigenvector is given by $\boldsymbol{\xi} = (1\,,1)^T$. Hence one solution is

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} t^{-3}\,.$$

By analogy with the scalar case considered in Section 5.4 and Example 2 of this section, we seek a second solution of the form $\mathbf{x} = \boldsymbol{\eta} t^{-3} \ln t + \boldsymbol{\zeta} t^{-3}$. Substituting these expressions into the differential equation we find that $\boldsymbol{\eta}$ and $\boldsymbol{\zeta}$ satisfy the equations $(\mathbf{A} + 3\mathbf{I})\boldsymbol{\eta} = \mathbf{0}$ and $(\mathbf{A} + 3\mathbf{I})\boldsymbol{\zeta} = \boldsymbol{\eta}$. Thus $\boldsymbol{\eta} = (1, 1)^T$ from above and $\boldsymbol{\zeta}$ then satisfies $4\zeta_1 - 4\zeta_2 = 1$, Choosing $\zeta_1 = 0$ we obtain $\zeta_2 = -1/4$ and hence a second solution is

$$\mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} t^{-3} \ln t + \begin{pmatrix} 0 \\ -1/4 \end{pmatrix} t^{-3}.$$

15. The characteristic equation is

$$r^2 - (a + d)r + ad - bc = 0\,.$$

Hence the eigenvalues are

$$r_{1,2} = \frac{a + d}{2} \pm \frac{1}{2}\sqrt{(a + d)^2 - 4(ad - bc)}\,.$$

Now if $ad - bc < 0$, then we have two real eigenvalues, one of which is positive. Thus there are solutions which do not converge to zero as $t \to \infty$. Also, if $ad - bc = 0$, then one of the eigenvalues is zero, so there are constant solutions other than zero, and these do not converge to zero as $t \to \infty$. So $ad - bc > 0$ is a necessary condition. When $ad - bc > 0$, we have two possibilities: either we have real solutions, or complex ones. In the complex case, the real part of the eigenvalues is $(a + d)/2$, so in order for all solutions to converge to zero, we need that $a + d < 0$. In the real case, we have two eigenvalues of the same sign, so again we need that $a + d < 0$. Thus the two constraints are necessary, and clearly they are also sufficient for all solutions to converge to zero as $t \to \infty$.

17.(a) We see that

$$(\mathbf{A} - 2\mathbf{I})^2 \boldsymbol{\eta} = (\mathbf{A} - 2\mathbf{I})(\mathbf{A} - 2\mathbf{I})\boldsymbol{\eta} = (\mathbf{A} - 2\mathbf{I})\boldsymbol{\xi} = \mathbf{0}.$$

(b) We compute:

$$(\mathbf{A} - 2\mathbf{I})^2 = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

(c) When $\boldsymbol{\eta} = (0, -1)^T$, then $(\mathbf{A} - 2\mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}$ gives us

$$\boldsymbol{\xi} = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \boldsymbol{\xi}^{(1)}.$$

(d) When $\boldsymbol{\eta} = (1, 0)^T$, then $(\mathbf{A} - 2\mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}$ gives us

$$\boldsymbol{\xi} = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = -\boldsymbol{\xi}^{(1)}.$$

(e) As in parts (c) and (d), when $\boldsymbol{\eta} = (k_1, k_2)^T$, then $(\mathbf{A} - 2\mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}$ gives us

$$\boldsymbol{\xi} = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} -k_1 - k_2 \\ k_1 + k_2 \end{pmatrix} = -(k_1 + k_2)\boldsymbol{\xi}^{(1)}.$$

18.(a) The eigensystem is obtained from analysis of the equation

$$\begin{pmatrix} 1-r & 1 & 1 \\ 2 & 1-r & -1 \\ -3 & 2 & 4-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The characteristic equation of the coefficient matrix is $-r^3 + 6r^2 - 12r + 8 = (2 - r)^3 = 0$, with the single root $r = 2$. Setting $r = 2$, we have

$$\begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -3 & 2 & 2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This system is reduced to the equations $\xi_1 - \xi_2 - \xi_3 = 0$ and $\xi_2 + \xi_3 = 0$, so the only eigenvectors are the multiples of $\boldsymbol{\xi}^{(1)} = (0, 1, -1)^T$.

(b) From part (a), one solution of the given differential equation is

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{2t}.$$

(c) Differentiating $\mathbf{x} = \boldsymbol{\xi} t e^{2t} + \boldsymbol{\eta} e^{2t}$ and using the fact that we want $\mathbf{x}' = \mathbf{A}\mathbf{x}$, we need $\boldsymbol{\xi} e^{2t} + 2\boldsymbol{\xi} t e^{2t} + 2\boldsymbol{\eta} e^{2t} = \mathbf{A}(\boldsymbol{\xi} t e^{2t} + \boldsymbol{\eta} e^{2t})$. Dividing by the nonzero term e^{2t} and rearranging, we obtain $\boldsymbol{\xi} = (\mathbf{A} - 2\mathbf{I})\boldsymbol{\xi} t + (\mathbf{A} - 2\mathbf{I})\boldsymbol{\eta}$. Thus we want to solve $(\mathbf{A} - 2\mathbf{I})\boldsymbol{\xi} = \mathbf{0}$, $(\mathbf{A} - 2\mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}$ with $\boldsymbol{\xi} = \boldsymbol{\xi}^{(1)}$ from part (a). This leads to the system of equations $\eta_1 - \eta_2 - \eta_3 = 0$ and $\eta_2 + \eta_3 = 1$, so if we choose $\eta_3 = 0$, then $\eta_1 = 1$ and $\eta_2 = 1$ and so $\boldsymbol{\eta} = (1, 1, 0)^T$. Hence a second solution of the equation is

$$\mathbf{x}^{(2)}(t) = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} t e^{2t} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{2t}.$$

(d) Assuming $\mathbf{x} = \boldsymbol{\xi}(t^2/2)e^{2t} + \boldsymbol{\eta} t e^{2t} + \boldsymbol{\zeta} e^{2t}$, we have $\mathbf{A}\mathbf{x} = \mathbf{A}\boldsymbol{\xi}(t^2/2)e^{2t} + \mathbf{A}\boldsymbol{\eta} t e^{2t} + \mathbf{A}\boldsymbol{\zeta} e^{2t}$ and $\mathbf{x}' = \boldsymbol{\xi} t e^{2t} + 2\boldsymbol{\xi}(t^2/2)e^{2t} + \boldsymbol{\eta} e^{2t} + 2\boldsymbol{\eta} t e^{2t} + 2\boldsymbol{\zeta} e^{2t}$ and thus $(\mathbf{A} - 2\mathbf{I})\boldsymbol{\xi} = 0$, $(\mathbf{A} - 2\mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}$ and $(\mathbf{A} - 2\mathbf{I})\boldsymbol{\zeta} = \boldsymbol{\eta}$. Again, $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are as found previously and the last equation is equivalent to

$$\begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -3 & 2 & 2 \end{pmatrix} \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

By row reduction we find the system $\zeta_1 - \zeta_2 - \zeta_3 = -1$ and $\zeta_2 + \zeta_3 = 3$, so if we choose $\zeta_2 = 0$, then $\zeta_1 = 2$ and $\zeta_3 = 3$ and so $\boldsymbol{\zeta} = (2, 0, 3)^T$. Hence a third solution of the equation is

$$\mathbf{x}^{(3)}(t) = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} (t^2/2)e^{2t} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} te^{2t} + \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} e^{2t}.$$

(e) $\mathbf{\Psi}$ is the matrix with $\mathbf{x}^{(1)}(t)$ as the first column, $\mathbf{x}^{(2)}(t)$ as the second column and $\mathbf{x}^{(3)}(t)$ as the third column.

(f) Here

$$\mathbf{T} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ -1 & 0 & 3 \end{pmatrix},$$

and using row operations on $[\mathbf{T} \,|\, \mathbf{I}\,]$ or a computer algebra system we get

$$\mathbf{T}^{-1} = \begin{pmatrix} -3 & 3 & 2 \\ 3 & -2 & -2 \\ -1 & 1 & 1 \end{pmatrix},$$

so we obtain that

$$\mathbf{T}^{-1}\mathbf{AT} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} = \mathbf{J},$$

which is equivalent to Eq.(29) for this problem.

20.(a) We compute:

$$\mathbf{J}^2 = \mathbf{JJ} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{pmatrix},$$

$$\mathbf{J}^3 = \mathbf{JJ}^2 = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{pmatrix} = \begin{pmatrix} \lambda^3 & 3\lambda^2 \\ 0 & \lambda^3 \end{pmatrix},$$

$$\mathbf{J}^4 = \mathbf{JJ}^3 = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} \lambda^3 & 3\lambda \\ 0 & \lambda^3 \end{pmatrix} = \begin{pmatrix} \lambda^4 & 4\lambda^3 \\ 0 & \lambda^4 \end{pmatrix}.$$

(b) Assume that

$$\mathbf{J}^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}.$$

Then

$$\mathbf{J}^{n+1} = \mathbf{JJ}^n = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix} = \begin{pmatrix} \lambda^{n+1} & (n+1)\lambda^n \\ 0 & \lambda^{n+1} \end{pmatrix},$$

which is the desired result.

(c) From Eq.(23) of Section 7.7, we have

$$e^{\mathbf{J}t} = \mathbf{I} + \sum_{n=1}^{\infty} \frac{\mathbf{J}^n t^n}{n!} = \mathbf{I} + \sum_{n=1}^{\infty} \begin{pmatrix} \lambda^n t^n/n! & n\lambda^{n-1}t^n/n! \\ 0 & \lambda^n t^n/n! \end{pmatrix} =$$

$$= \begin{pmatrix} 1+\sum_{n=1}^{\infty}\lambda^n t^n/n! & t\sum_{n=1}^{\infty}\lambda^{n-1}t^{n-1}/(n-1)! \\ 0 & 1+\sum_{n=1}^{\infty}\lambda^n t^n/n! \end{pmatrix} = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}.$$

(d) From Eq.(28) of Section 7.7, we have

$$\mathbf{x} = e^{\mathbf{J}t}\mathbf{x}^0 = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}\begin{pmatrix} \mathbf{x}_1^0 \\ \mathbf{x}_2^0 \end{pmatrix} = \begin{pmatrix} \mathbf{x}_1^0 e^{\lambda t} + \mathbf{x}_2^0 te^{\lambda t} \\ \mathbf{x}_2^0 e^{\lambda t} \end{pmatrix} = \begin{pmatrix} \mathbf{x}_1^0 \\ \mathbf{x}_2^0 \end{pmatrix} e^{\lambda t} + \begin{pmatrix} \mathbf{x}_2^0 \\ 0 \end{pmatrix} te^{\lambda t}.$$

7.9

1. The eigenvalues of

$$\begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}$$

are given by $r_1 = 1$ and $r_2 = -1$. Corresponding eigenvectors are given by

$$\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

Therefore, two linearly independent solutions are given by

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t}, \qquad \mathbf{x}^{(2)}(t) = \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t},$$

and

$$\boldsymbol{\Psi}(t) = \begin{pmatrix} e^{t} & e^{-t} \\ e^{t} & 3e^{-t} \end{pmatrix}$$

is a fundamental matrix. In order to find the general solution using variation of parameters, we need to calculate $\int_{t_1}^{t} \boldsymbol{\Psi}^{-1}(s)\mathbf{g}(s)ds$. We see that

$$\boldsymbol{\Psi}^{-1}(s) = \frac{1}{2}\begin{pmatrix} 3e^{-s} & -e^{-s} \\ -e^{s} & e^{s} \end{pmatrix}.$$

Therefore,

$$\int_{t_1}^{t} \boldsymbol{\Psi}^{-1}(s)\mathbf{g}(s)\,ds = \frac{1}{2}\int_{t_1}^{t} \begin{pmatrix} 3e^{-s} & -e^{-s} \\ -e^{s} & e^{s} \end{pmatrix}\begin{pmatrix} e^{s} \\ s \end{pmatrix} ds =$$

$$= \frac{1}{2}\int_{t_1}^{t} \begin{pmatrix} 3 - se^{-s} \\ -e^{2s} + se^{s} \end{pmatrix} ds = \frac{1}{2}\begin{pmatrix} 3t + te^{-t} + e^{-t} \\ -\frac{1}{2}e^{2t} + te^{t} - e^{t} \end{pmatrix} + \mathbf{c}.$$

Then the general solution will be given by

$$\begin{aligned} \mathbf{x}(t) &= \boldsymbol{\Psi}(t)\mathbf{c} + \boldsymbol{\Psi}(t)\int_{t_1}^{t} \boldsymbol{\Psi}^{-1}(s)\mathbf{g}(s)\,ds \\ &= \begin{pmatrix} e^{t} & e^{-t} \\ e^{t} & 3e^{-t} \end{pmatrix}\mathbf{c} + \begin{pmatrix} e^{t} & e^{-t} \\ e^{t} & 3e^{-t} \end{pmatrix}\left[\frac{1}{2}\begin{pmatrix} 3t + te^{-t} + e^{-t} \\ -\frac{1}{2}e^{2t} + te^{t} - e^{t} \end{pmatrix} + \mathbf{c}\right] \\ &= c_1 e^{t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 1 \\ 3 \end{pmatrix} + \begin{pmatrix} \left(\frac{3}{2}t - \frac{1}{4}\right)e^{t} + t \\ \left(\frac{3}{2}t - \frac{3}{4}\right)e^{t} + 2t - 1 \end{pmatrix}. \end{aligned}$$

2. For this problem, we illustrate the use of the Laplace Transform. As in Eq.(43),

$$(s\mathbf{I}-\mathbf{A})\mathbf{X} = \begin{pmatrix} \dfrac{1}{s-1} \\ \dfrac{\sqrt{3}}{s+1} \end{pmatrix},$$

where we have assumed that $\mathbf{x}(0)=0$. We find that

$$s\mathbf{I}-\mathbf{A} = s\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} = \begin{pmatrix} s-1 & -\sqrt{3} \\ -\sqrt{3} & s+1 \end{pmatrix},$$

which implies that

$$(s\mathbf{I}-\mathbf{A})^{-1} = \frac{1}{s^2-4}\begin{pmatrix} 1+s & \sqrt{3} \\ \sqrt{3} & s-1 \end{pmatrix},$$

which gives

$$\mathbf{X}(s) = (s\mathbf{I}-\mathbf{A})^{-1}\begin{pmatrix} \dfrac{1}{s-1} \\ \dfrac{\sqrt{3}}{s+1} \end{pmatrix} = \begin{pmatrix} \dfrac{1}{s-2} - \dfrac{2/3}{s-1} - \dfrac{1}{s+1} + \dfrac{2/3}{s+2} \\ \dfrac{\sqrt{3}/3}{s-2} - \dfrac{\sqrt{3}/3}{s-1} + \dfrac{2\sqrt{3}/3}{s+1} - \dfrac{2\sqrt{3}/3}{s+2} \end{pmatrix},$$

using partial fractions. The inverse transform then gives

$$x(t) = \begin{pmatrix} 1 \\ \sqrt{3}/3 \end{pmatrix} e^{2t} - \frac{1}{3}\begin{pmatrix} 2 \\ \sqrt{3} \end{pmatrix} e^{t} + \begin{pmatrix} -1 \\ 2\sqrt{3}/3 \end{pmatrix} e^{-t} + \frac{2}{3}\begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} e^{-2t}.$$

As in Example 4, in order to obtain the general solution of the differential equation, we must add the general solution of the homogeneous system to this particular solution. This particular solution differs from the one given in the text by a multiple of the homogeneous solution.

3. The eigenvalues of

$$\begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}$$

are given by $r_1 = i$ and $r_2 = -i$. For $r = i$, $\boldsymbol{\xi} = (2+i, 1)^T$ is a corresponding eigenvector, and

$$\mathbf{x}(t) = e^{it}\begin{pmatrix} 2+i \\ 1 \end{pmatrix}$$

is a solution of the homogeneous equation. Looking at the real and imaginary parts of $\mathbf{x}$, we have the following two linearly independent, real-valued solutions of the homogeneous equation:

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}\cos t - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\sin t, \quad \mathbf{x}^{(2)}(t) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}\sin t + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\cos t.$$

Therefore,

$$\boldsymbol{\Psi}(t) = \begin{pmatrix} 2\cos t - \sin t & 2\sin t + \cos t \\ \cos t & \sin t \end{pmatrix}$$

is a fundamental matrix. In order to calculate the general solution, we need to calculate $\int_{t_1}^t \mathbf{\Psi}^{-1}(s)\mathbf{g}(s)ds$. We see that

$$\mathbf{\Psi}^{-1}(s) = \begin{pmatrix} -\sin s & 2\sin s + \cos s \\ \cos s & -2\cos s + \sin s \end{pmatrix}.$$

Therefore,

$$\int_{t_1}^t \mathbf{\Psi}^{-1}(s)\mathbf{g}(s)\,ds = \int_{t_1}^t \begin{pmatrix} -\sin s & 2\sin s + \cos s \\ \cos s & -2\cos s + \sin s \end{pmatrix} \begin{pmatrix} -\cos s \\ \sin s \end{pmatrix} ds =$$

$$= \int_{t_1}^t \begin{pmatrix} 2\sin s\cos s + 2\sin^2 s \\ -2\cos s\sin s + \sin^2 s - \cos^2 s \end{pmatrix} ds = \int_{t_1}^t \begin{pmatrix} \sin 2s + 1 - \cos 2s \\ -\sin 2s - \cos 2s \end{pmatrix} ds =$$

$$= \frac{1}{2}\begin{pmatrix} -\cos 2t + 2t - \sin 2t \\ \cos 2t - \sin 2t \end{pmatrix} + \mathbf{c}.$$

Then the general solution will be given by

$$\mathbf{x}(t) = \mathbf{\Psi}(t)\mathbf{c} + \mathbf{\Psi}(t)\int_{t_1}^t \mathbf{\Psi}^{-1}(s)\mathbf{g}(s)\,ds = \begin{pmatrix} 2\cos t - \sin t & 2\sin t + \cos t \\ \cos t & \sin t \end{pmatrix}\mathbf{c}$$

$$+ \frac{1}{2}\begin{pmatrix} 2\cos t - \sin t & 2\sin t + \cos t \\ \cos t & \sin t \end{pmatrix}\begin{pmatrix} -\cos 2t + 2t - \sin 2t \\ \cos 2t - \sin 2t \end{pmatrix}$$

$$= c_1\begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + c_2\begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 4t\cos t - 2t\sin t - \cos t - 3\sin t \\ 2t\cos t - \cos t - \sin t \end{pmatrix}.$$

We note that in multiplying the last two matrices, we made use of the trigonometric identities, $\cos 2t = 2\cos^2 t - 1$ and $\cos 2t = 1 - 2\sin^2 t$.

4. In this problem we use the method illustrated in Example 1. From Problem 4 of Section 7.5 we have the transformation matrix

$$\mathbf{T} = \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix}.$$

Inverting $\mathbf{T}$ we find that

$$\mathbf{T}^{-1} = \frac{1}{5}\begin{pmatrix} 1 & -1 \\ 4 & 1 \end{pmatrix}.$$

If we let $\mathbf{x} = \mathbf{T}\mathbf{y}$ and substitute into the differential equation, we obtain

$$\mathbf{y}' = \frac{1}{5}\begin{pmatrix} 1 & -1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix}\mathbf{y} + \frac{1}{5}\begin{pmatrix} 1 & -1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} e^{-2t} \\ -2e^t \end{pmatrix} =$$

$$\begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix}\mathbf{y} + \frac{1}{5}\begin{pmatrix} e^{-2t} + 2e^t \\ 4e^{-2t} - 2e^t \end{pmatrix}.$$

This corresponds to the two scalar equations $y_1' + 3y_1 = (1/5)e^{-2t} + (2/5)e^t$ and $y_2' - 2y_2 = (4/5)e^{-2t} - (2/5)e^t$, which may be solved by the methods of Section 2.1. For the first equation the integrating factor is e^{3t} and we obtain $(e^{3t}y_1)' = (1/5)e^t + (2/5)e^{4t}$, so $e^{3t}y_1 = (1/5)e^t + (1/10)e^{4t} + c_1$. For the second equation the integrating factor is e^{-2t}, so $(e^{-2t}y_2)' = (4/5)e^{-4t} - (2/5)e^{-t}$, hence $e^{-2t}y_2 = -(1/5)e^{-4t} + (2/5)e^{-t} + c_2$. Thus

$$\mathbf{y} = \begin{pmatrix} 1/5 \\ -1/5 \end{pmatrix} e^{-2t} + \begin{pmatrix} 1/10 \\ 2/5 \end{pmatrix} e^{t} + \begin{pmatrix} c_1 e^{-3t} \\ c_2 e^{2t} \end{pmatrix}.$$

Finally, multiplying by $\mathbf{T}$, we obtain

$$\mathbf{x} = \mathbf{T}\mathbf{y} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{-2t} + \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} e^{t} + c_1 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}.$$

The last two terms are the general solution of the corresponding homogeneous system, while the first two terms constitute a particular solution of the nonhomogeneous system.

8. The eigenvalues of the coefficient matrix are $r_1 = 1$ and $r_2 = -1$. It follows that the solution of the homogeneous equation is

$$\mathbf{x}_c = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t}.$$

Use the method of undetermined coefficients. Since the right hand side is related to one of the fundamental solutions, set $\mathbf{v}{=}\mathbf{a}\, te^t{+}\mathbf{b}\, e^t$. Substitution into the ODE yields

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} (e^t + te^t) + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} e^t = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} te^t + \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} e^t + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t.$$

In scalar form, we have

$$\begin{aligned} (a_1 + b_1)e^t + a_1 te^t &= (2a_1 - a_2)te^t + (2b_1 - b_2)e^t + e^t \\ (a_2 + b_2)e^t + a_2 te^t &= (3a_1 - 2a_2)te^t + (3b_1 - 2b_2)e^t - e^t\,. \end{aligned}$$

Equating the coefficients in these two equations, we find that

$$\begin{aligned} a_1 &= 2a_1 - a_2 \\ a_1 + b_1 &= 2b_1 - b_2 + 1 \\ a_2 &= 3a_1 - 2a_2 \\ a_2 + b_2 &= 3b_1 - 2b_2 - 1\,. \end{aligned}$$

It follows that $a_1 = a_2$. Setting $a_1 = a_2 = a$, the equations reduce to

$$\begin{aligned} b_1 - b_2 &= a - 1 \\ 3b_1 - 3b_2 &= 1 + a\,. \end{aligned}$$

Combining these equations, it is necessary that $a = 2$. As a result, $b_1 = b_2 + 1$. Choosing $b_2 = k$, some arbitrary constant, a particular solution is

$$\mathbf{v} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} te^t + \begin{pmatrix} k+1 \\ k \end{pmatrix} e^t = \begin{pmatrix} 2 \\ 2 \end{pmatrix} te^t + k \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t.$$

Since the second vector is a fundamental solution, the general solution can be written as

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t} + \begin{pmatrix} 2 \\ 2 \end{pmatrix} te^t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t.$$

12. Since the coefficient matrix is the same as that of Problem 3, use the same procedure as done in that problem, including the $\mathbf{\Psi}^{-1}$ found there. In the interval $\pi/2 < t < \pi$, $\sin t > 0$ and $\cos t < 0$, hence $|\sin t| = \sin t$ but $|\cos t| = -\cos t$.

14. The general solution of the homogeneous problem is

$$\mathbf{x}^{(c)} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} t + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} t^{-1},$$

which can be verified by substitution into the system of ODEs. Since the vectors are linearly independent, a fundamental matrix is given by

$$\mathbf{\Psi}(t) = \begin{pmatrix} t & t^{-1} \\ t & 3t^{-1} \end{pmatrix}.$$

The inverse of the fundamental matrix is

$$\mathbf{\Psi}^{-1}(t) = \frac{1}{2} \begin{pmatrix} 3t^{-1} & -t^{-1} \\ -t & t \end{pmatrix}.$$

From the given problem statement

$$\mathbf{g}(t) = \begin{pmatrix} t^{-1} - t \\ 2 \end{pmatrix}$$

after dividing both sides by $t \neq 0$. Proceeding with the method of variation of parameters,

$$\mathbf{\Psi}^{-1}(t)\mathbf{g}(t) = \begin{pmatrix} -3/2 - t^{-1} + (3/2)t^{-2} \\ -1/2 + t - t^2/2 \end{pmatrix},$$

and

$$\mathbf{u} = \int \mathbf{\Psi}^{-1}(t)\mathbf{g}(t)\, dt = \begin{pmatrix} -(3/2)t - \ln t - (3/2)t^{-1} + c_1 \\ -t/2 + t^2/2 - t^3/6 + c_2 \end{pmatrix}.$$

Multiplication of $\mathbf{u}$ by $\mathbf{\Psi}(t)$ yields the desired solution.