CHAPTER

5

Series Solutions of Second Order Linear Equations

5.1

2. Use the ratio test:

$$\lim_{n\to\infty} \frac{\left|(n+1)x^{n+1}/2^{n+1}\right|}{|nx^n/2^n|} = \lim_{n\to\infty} \frac{n+1}{n}\frac{1}{2}|x| = \frac{|x|}{2}.$$

Therefore the series converges absolutely for $|x| < 2$. For $x = 2$ and $x = -2$ the n^{th} term does not approach zero as $n \to \infty$ so the series diverge. Hence the radius of convergence is $\rho = 2$.

5. Use the ratio test :

$$\lim_{n\to\infty} \frac{\left|(2x+1)^{n+1}/(n+1)^2\right|}{|(2x+1)^n/n^2|} = \lim_{n\to\infty} \frac{n^2}{(n+1)^2}|2x+1| = |2x+1|.$$

Therefore the series converges absolutely for $|2x+1| < 1$, or $|x+1/2| < 1/2$. The radius of convergence is $\rho = 1/2$. At $x = 0$ and $x = -1$, the series also converges absolutely.

9. For this problem $f(x) = \sin x$, so $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$, $f''''(x) = \sin x \ldots$, and thus $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, $f'''(0) = -1 \ldots$. The even terms in the series vanish and the odd terms alternate in sign. We obtain that $\sin x = \sum_{n=0}^{\infty}(-1)^n x^{2n+1}/(2n+1)!$. Also,

$$\lim_{n\to\infty} \frac{\left|(-1)^{n+1}x^{2n+3}/(2n+3)!\right|}{|(-1)^n x^{2n+1}/(2n+1)!|} = \lim_{n\to\infty} x^2 \frac{1}{(2n+3)(2n+2)} = 0,$$

so the series converges for all x and $\rho = \infty$.

12. For this problem $f(x) = x^2$. Hence $f'(x) = 2x$, $f''(x) = 2$, and $f^{(n)}(x) = 0$ for $n > 2$. Then $f(-1) = 1$, $f'(-1) = -2$, $f''(-1) = 2$ and $x^2 = 1 - 2(x+1) + 2(x+1)^2/2! = 1 - 2(x+1) + (x+1)^2$. Since the series terminates after a finite number of terms, it converges for all x. Thus $\rho = \infty$.

13. For this problem $f(x) = \ln x$. Hence $f'(x) = 1/x$, $f''(x) = -1/x^2$, $f''(x) = 1 \cdot 2/x^3 \ldots$, and $f^{(n)}(x) = (-1)^{n+1}(n-1)!/x^n$. Then $f(1) = 0$, $f'(1) = 1$, $f''(1) = -1$, $f'''(1) = 1 \cdot 2 \ldots$, $f^{(n)}(1) = (-1)^{n+1}(n-1)!$. The Taylor series is

$$\ln x = (x-1) - (x-1)^2/2 + (x-1)^3/3 - \ldots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}.$$

It follows from the ratio test that the series converges absolutely for $|x-1| < 1$. Thus $\rho = 1$. The series diverges at $x = 0$ and converges at $x = 2$.

18. We have $y = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$, so $y' = a_1 + 2a_2 x + 3a_3 x^2 + \ldots + (n+1)a_{n+1}x^n + \ldots$, and then we get $y'' = 2a_2 + 3 \cdot 2a_3 x + 4 \cdot 3a_4 x^2 + \ldots + (n+2)(n+1)a_{n+2}x^n + \ldots$. If $y'' = y$, we equate coefficients of like powers of x to obtain $2a_2 = a_0$, $3 \cdot 2a_3 = a_1$, $4 \cdot 3a_4 = a_2, \ldots (n+2)(n+1)a_{n+2} = a_n$. Thus $a_2 = a_0/2, a_3 = a_1/6, a_4 = a_2/(4 \cdot 3) = a_0/4!, \ldots a_{n+2} = a_n/((n+2)(n+1))$. These yield the desired results for $n = 0, 1, 2, 3, \ldots$

19. Set $m = n - 1$ on the right hand side of the equation. Then $n = m + 1$ and when $n = 1$, $m = 0$. Thus the right hand side becomes $\sum_{m=0}^{\infty} a_m (x-1)^{m+1}$, which is the same as the left hand side when m is replaced by n.

23. Multiplying each term of the first series by x yields

$$x \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^n = \sum_{n=0}^{\infty} n a_n x^n,$$

where the last equality comes from $n a_n = 0$ for $n = 0$. Changing the index from k to n in the second series yields

$$\sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (n+1) a_n x^n.$$

25.

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} + x \sum_{k=1}^{\infty} k a_k x^{k-1} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{k=1}^{\infty} k a_k x^k =$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} + n a_n\right] x^n.$$

In the first case we have let $n = m - 2$ in the first summation and multiplied each term of the second summation by x. In the second case we have let $n = k$ and noted that for $n = 0$, $n a_n = 0$.

28. If we shift the index of summation in the first sum by letting $m = n-1$, we have

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{m=0}^{\infty} (m+1) a_{m+1} x^m.$$

Substituting this into the given equation and letting $m = n$ again, we obtain

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0, \text{ or } \sum_{n=0}^{\infty} \left[(n+1)a_{n+1} + 2a_n\right] x^n = 0.$$

Hence $a_{n+1} = -2a_n/(n+1)$ for $n = 0, 1, 2, 3, ...$ Thus $a_1 = -2a_0$, $a_2 = -2a_1/2 = 2^2 a_0/2$, $a_3 = -2a_2/3 = -2^3 a_0/2\cdot 3 = -2^3 a_0/3!...$ and $a_n = (-1)^n 2^n a_0/n!$. Notice that for $n = 0$ this formula reduces to a_0, so we can write

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (-1)^n 2^n a_0 x^n/n! = a_0 \sum_{n=0}^{\infty} (-2x)^n/n! = a_0 e^{-2x}.$$

5.2

2.(a) $y = \sum_{n=0}^{\infty} a_n x^n; y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$ and since we must multiply y' by x in the D.E. we do not shift the index; and $y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$. Substituting in the D.E., we obtain

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0.$$

In order to have the starting point the same in all three summations, we let $n = 0$ in the first and third terms to obtain the following:

$$(2\cdot 1 a_2 - a_0)x^0 + \sum_{n=1}^{\infty} \left[(n+2)(n+1)a_{n+2} - (n+1)a_n\right] x^n = 0.$$

Thus $a_{n+2} = a_n/(n+2)$ for $n = 1, 2, 3, \ldots$ Note that the recurrence relation is also correct for $n = 0$.

(b) From the recurrence relation we have $a_2 = a_0/2$, $a_4 = a_2/4 = a_0/2\cdot 4$, $a_6 = a_4/6 = a_0/2\cdot 4\cdot 6$, so $y_1 = 1 + x^2/2 + x^4/2\cdot 4 + x^6/2\cdot 4\cdot 6 + \ldots$, and $a_3 = a_1/3$, $a_5 = a_3/5 = a_1/3\cdot 5$, $a_7 = a_5/7 = a_1/3\cdot 5\cdot 7$, so $y_2 = x + x^3/3 + x^5/3\cdot 5 + x^7/3\cdot 5\cdot 7 + \ldots$

(c) $W(y_1, y_2)(0) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$ and thus y_1, y_2 form a fundamental set of solutions.

(d) From part (b) we see the even coefficients can be written as $a_{2m} = a_0/2^m m!$. For the odd coefficients notice that $a_3 = 2a_1/(2\cdot 3) = 2a_1/3!$, that $a_5 = 2\cdot 4 a_1/(2\cdot 3\cdot$

$4 \cdot 5) = 2^2 \cdot 2a_1/5!$, and that $a_7 = 2 \cdot 4 \cdot 6a_1/(2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7) = 2^3 \cdot 3!a_1/7!$. Likewise $a_9 = a_7/9 = 2^3 \cdot 3!a_1/(7!)9 = 2^3 \cdot 3!8a_1/9! = 2^4 \cdot 4!a_1/9!$. Continuing, we have $a_{2m+1} = 2^m m!a_1/(2m+1)!$. Thus

$$y = a_0 \sum_{m=0}^{\infty} \frac{x^{2m}}{2^m m!} + a_1 \sum_{m=0}^{\infty} \frac{2^m m! x^{2m+1}}{(2m+1)!}.$$

3.(a)

$$y = \sum_{n=0}^{\infty} a_n (x-1)^n; \;\; y' = \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} (x-1)^n,$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n.$$

Substituting in the differential equation and setting $x = 1 + (x-1)$, we obtain

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n - \sum_{n=0}^{\infty} (n+1) a_{n+1} (x-1)^n - \sum_{n=1}^{\infty} n a_n (x-1)^n$$
$$- \sum_{n=0}^{\infty} a_n (x-1)^n = 0,$$

where the third sum comes from

$$-(x-1)y' = -\sum_{n=0}^{\infty} (n+1) a_{n+1} (x-1)^{n+1} = -\sum_{n=1}^{\infty} n a_n (x-1)^n.$$

Letting $n = 0$ in the first, second and fourth sums we obtain

$$(2 \cdot 1 \cdot a_2 - 1 \cdot a_1 - a_0)(x-1)^0 +$$
$$+ \sum_{n=1}^{\infty} \left[(n+2)(n+1) a_{n+2} - (n+1) a_{n+1} - (n+1) a_n \right] (x-1)^n = 0.$$

Setting the terms in the square brackets equal to zero and dividing by $(n+1)$ gives us that $(n+2)a_{n+2} - a_{n+1} - a_n = 0$ for $n = 1, 2, 3, \ldots$ (which also holds for $n = 0$). This recurrence relation can be used to solve for a_2 in terms of a_0 and a_1, then for a_3 in terms of a_1 and a_2 and so on.

(b) In many cases it is easier to first take $a_0 = 0$ and generate one solution and then take $a_1 = 0$ and generate a second solution. Thus, choosing $a_0 = 0$, we find that $a_2 = a_1/2$, $a_3 = (a_2 + a_1)/3 = a_1/2$, $a_4 = (a_3 + a_2)/4 = a_1/4$, $a_5 = (a_4 + a_3)/5 = 3a_1/20, \ldots$. This yields the solution $y_2(x) = (x-1) + (x-1)^2/2 + (x-1)^3/2 + (x-1)^4/4 + \ldots$. The second solution may be obtained by choosing $a_1 = 0$. Then $a_2 = a_0/2$, $a_3 = (a_2 + a_1)/3 = a_0/6$, $a_4 = (a_3 + a_2)/4 = a_0/6$, $a_5 = (a_4 + a_3)/5 = a_0/15, \ldots$. This yields the solution $y_1(x) = 1 + (x-1)^2/2 + (x-1)^3/6 + (x-1)^4/6 + (x-1)^5/15 + \ldots$.

(c) $W(y_1, y_2)(1) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$ and thus y_1 and y_2 form a fundamental set of solutions.

(d) A general term is not easily found in this case.

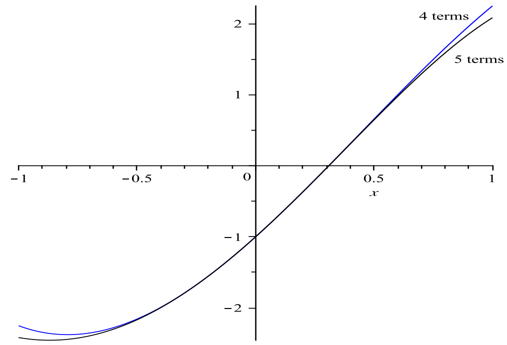
5.
$$y = \sum_{n=0}^{\infty} a_n x^n; \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}; \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$
Substituting in the differential equation and shifting the index in both summations for y'' gives
$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty}(n+1)na_{n+1}x^n + \sum_{n=0}^{\infty} a_n x^n = (2 \cdot 1 \cdot a_2 + a_0)x^0$$
$$+\sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2} - (n+1)na_{n+1} + a_n\right]x^n = 0.$$
Thus $a_2 = -a_0/2$ and $a_{n+2} = na_{n+1}/(n+2) - a_n/(n+2)(n+1)$, $n = 1, 2, \ldots$. Choosing $a_0 = 0$ yields $a_2 = 0$, $a_3 = -a_1/6$, $a_4 = 2a_3/4 = -a_1/12$, $a_5 = 3a_4/5 - a_3/20 = -a_1/24, \ldots$, and hence $y_2(x) = a_1(x - x^3/6 - x^4/12 - x^5/24 + \ldots)$. A second linearly independent solution is obtained by choosing $a_1 = 0$. Then $a_2 = -a_0/2$, $a_3 = a_2/3 = -a_0/6$, $a_4 = 2a_3/4 - a_2/12 = -a_0/24, \ldots$, which gives $y_1(x) = a_0(1 - x^2/2 - x^3/6 - x^4/24 + \ldots)$.

8. If $y = \sum_{n=0}^{\infty} a_n(x-1)^n$, then
$$xy = [1 + (x-1)]y = \sum_{n=0}^{\infty} a_n(x-1)^n + \sum_{n=0}^{\infty} a_n(x-1)^{n+1},$$
$$y' = \sum_{n=1}^{\infty} na_n(x-1)^{n-1}, \text{ and}$$
$$xy'' = [1 + (x-1)]y'' = \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-1}.$$

14. We need to rewrite $x+1$ as $3 + (x-2)$ in order to multiply $x+1$ times y' as a power series about $x_0 = 2$.

16.(a) From Problem 6 we have $y(x) = c_1(1 - x^2 + x^4/6 + ...) + c_2(x - x^3/4 + 7x^5/160 + ...)$. Now $y(0) = c_1 = -1$ and $y'(0) = c_2 = 3$ and thus
$$\begin{aligned} y(x) &= -1 + x^2 - \frac{1}{6}x^4 + ... + 3x - \frac{3}{4}x^3 + ... \\ &= -1 + 3x + x^2 - \frac{3}{4}x^3 - \frac{1}{6}x^4 + ... \end{aligned}$$

(b)



(c) It appears that f is a reasonable approximation for $|x| < 0.7$. In fact, the magnitude of the difference in the two graphs is 0.02 for $|x| = 0.6$ and 0.04 for $|x| = 0.7$.

19. Letting $t = x - 1$ yields $(x-1)^2 = t^2$ and $(x^2 - 1) = t^2 + 2t$. Now let $u(t) = y(t+1)$ and hence $u' = y'$ and $u'' = y''$. Thus the differential equation transforms into $u''(t) + t^2 u'(t) + (t^2 + 2t)u(t) = 0$. Assuming that $u(t) = \sum_{n=0}^{\infty} a_n t^n$, we have $u'(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$ and $u''(t) = \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2}$. Substituting in the differential equation and shifting indices yields

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n + \sum_{n=2}^{\infty}(n-1)a_{n-1}t^n + \sum_{n=2}^{\infty} a_{n-2}t^n + \sum_{n=1}^{\infty} 2a_{n-1}t^n = 0,$$

$$2 \cdot 1 \cdot a_2 t^0 + (3 \cdot 2 \cdot a_3 + 2 \cdot a_0)t^1 +$$

$$+\sum_{n=2}^{\infty}\left[(n+2)(n+1)a_{n+2} + (n+1)a_{n-1} + a_{n-2}\right]t^n = 0.$$

It follows that $a_2 = 0$, $a_3 = -a_0/3$, and $a_{n+2} = -a_{n-1}/(n+2) - a_{n-2}/(n+2)(n+1)$, $n = 2, 3, 4, \ldots$. We obtain one solution by choosing $a_1 = 0$. Then $a_4 = -a_0/12$, $a_5 = -a_2/5 - a_1/20 = 0$, $a_6 = -a_3/6 - a_2/30 = a_0/18, \ldots$. Thus one solution is $u_1(t) = a_0(1 - t^3/3 - t^4/12 + t^6/18 + \ldots)$, so $y_1(x) = u_1(x-1) = 1 - (x-1)^3/3 - (x-1)^4/12 + (x-1)^6/18 + \ldots$. We obtain a second solution by choosing $a_0 = 0$. Then $a_4 = -a_1/4$, $a_5 = -a_2/5 - a_1/20 = -a_1/20$, $a_6 = -a_3/6 - a_2/30 = 0$, $a_7 = -a_4/7 - a_3/42 = a_1/28, \ldots$. Thus $u_2(t) = t - t^4/4 - t^5/20 + t^7/28 + \ldots$, or

$$y_2(x) = u_2(x-1) = (x-1) - (x-1)^4/4 - (x-1)^5/20 + (x-1)^7/28 + \ldots.$$

The Taylor series for $x^2 - 1$ about $x = 1$ may be obtained by writing $x = 1 + (x-1)$ so $x^2 = 1 + 2(x-1) + (x-1)^2$ and $x^2 - 1 = 2(x-1) + (x-1)^2$. The differential equation now appears as $y'' + (x-1)^2 y' + \left[(x-1)^2 + 2(x-1)\right] y = 0$, which is identical to the transformed equation with $t = x - 1$.

22.(a) Clearly, $(\sin x)' = \cos x = \sqrt{1 - \sin^2 x}$ (for $-\pi/2 \le x \le \pi/2$) and $\sin 0 = 0$.

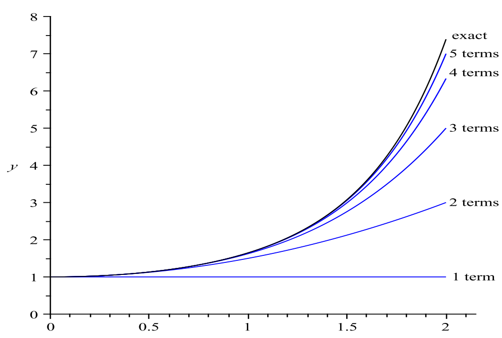
(b) $y = a_0 + a_1x + a_2x^2 + \ldots$, $y^2 = a_0^2 + 2a_0a_1x + \left(2a_0a_2 + a_1^2\right)x^2 + \ldots$, $y' = a_1 + 2a_2x + 3a_3x^2 + \ldots$, and $(y')^2 = a_1^2 + 4a_1a_2x + \left(6a_1a_3 + 4a_2^2\right)x^2 + \ldots$. Substituting these into $(y')^2 = 1 - y^2$ and collecting coefficients of like powers of x yields

$$\left(a_1^2 + a_0^2 - 1\right) + \left(4a_1a_2 + 2a_0a_1\right)x + \left(6a_1a_3 + 4a_2^2 + 2a_0a_2 + a_1^2\right)x^2 + \ldots = 0.$$

As in the earlier problems, each coefficient must be zero. The I.C. $y(0) = 0$ requires that $a_0 = 0$, and thus $a_1^2 + a_20 - 1 = 0$ gives $a_1^2 = 1$. However, the D.E. indicates that y' is always positive, so $y'(0) = a_1 > 0$ implies $a_1 = 1$. Then $4a_1a_2 + 2a_0a_1 = 0$ implies that $a_2 = 0$; and $6a_1a_3 + 4a_2^2 + 2a_0a_2 + a_21 = 6a_1a_3 + a_1^2 = 0$ implies that $a_3 = -1/6$. Thus $y = x - x^3/3! + \ldots$, which are the first two terms of the Taylor series or $\sin x$.

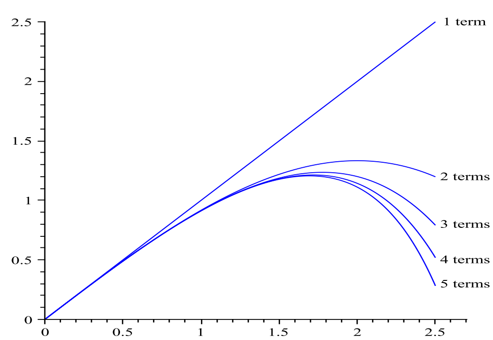
23. With the given initial conditions, Problem 2 tells us that the series solution is given by

$$y(x) = 1 + \frac{1}{2}x^2 + \frac{1}{2^2\,2!}x^4 + \frac{1}{2^3 3!}x^6 + \frac{1}{2^4 4!}x^8 + \ldots .$$



26. With the given initial conditions, Problem 10 tells us that the series solution is given by

$$y(x) = x - \frac{x^3}{12} - \frac{x^5}{240} - \frac{x^7}{2240} - \frac{x^9}{16128} - \ldots .$$



5.3

1. The differential equation can be solved for y'' to yield $y'' = -xy' - y$. If $y = \phi(x)$ is a solution, then $\phi''(x) = -x\phi'(x) - \phi(x)$ and thus setting $x = 0$ we obtain that $\phi''(0) = -0 - 1 = -1$. Differentiating the equation for y'' yields $y''' = -xy'' - 2y'$ and hence setting $y = \phi(x)$ again yields $\phi'''(0) = -0 - 0 = 0$. In a similar fashion $y^{(4)} = -xy''' - 3y''$ and thus $\phi^{(4)}(0) = -0 - 3(-1) = 3$. The process can be continued to calculate higher derivatives of $\phi(x)$.

3. Let $y = \phi(x)$ be a solution of the initial value problem. First write

$$y'' = -\frac{1+x}{x^2}\,y' - \frac{3\ln x}{x^2}\,y\,.$$

Differentiating twice,

$$y''' = \frac{-1}{x^3}\left[(x+x^2)y'' + (3x\ln x - x - 2)y' + (3 - 6\ln x)y\right].$$

$$\begin{aligned} y^{(4)} &= \frac{-1}{x^4}\Big[(x^2+x^3)y''' + (3x^2\ln x - 2x^2 - 4x)y'' + \\ &\quad + (6 + 8x - 12x\ln x)y' + (18\ln x - 15)y\Big]. \end{aligned}$$

Given that $\phi(1) = 2$ and $\phi'(1) = 0$, the first equation gives $\phi''(1) = 0$ and the last two equations give $\phi'''(1) = -6$ and $\phi^{(4)}(1) = 42$.

6. The zeros of $P(x) = x^2 - 2x - 3$ are $x = -1$ and $x = 3$. For $x_0 = 4$, $x_0 = -4$, and $x_0 = 0$ the distance to the nearest zero of $P(x)$ is $1, 3,$ and 1, respectively. Thus a lower bound for the radius of convergence for series solutions in powers of $(x-4)$, $(x+4)$, and x is $\rho = 1$, $\rho = 3$, and $\rho = 1$, respectively.

7. The zeros of $P(x) = 1 + x^3$ are the three cube roots of -1. They all lie on the unit circle in the complex plane. So for $x_0 = 0$, $\rho = 1$. For $x_0 = 2$, the nearest root is $e^{i\pi/3} = (1 + i\sqrt{3})/2$, hence $\rho = \sqrt{3}$.

9.(a) Since $p(x) = 1$ has no zeros, the radius of convergence about $x_0 = 0$ is $\rho = \infty$.

(b) $p(x) = -x$ and $q(x) = -1$ are analytic for all x.

(c) $p(x) = -x$ and $q(x) = -1$ are analytic for all x.

(d) $p(x) = 0$ and $q(x) = kx^2$ are analytic for all x.

(e) The only root of $P(x) = 1 - x$ is 1. Hence $\rho = 1$.

(f) Since $P(x) = 2 + x^2$ has zeros at $x = \pm\sqrt{2}i$, the lower bound for the radius of convergence of the series solution about $x_0 = 0$ is $\rho = \sqrt{2}$.

(g) $p(x) = x$ and $q(x) = 2$ are analytic for all x.

(h) $P(x) = x$ has a zero at $x = 0$ and since $x_0 = 1$, $\rho = 1$.

(i) The zeroes of $P(x) = 1 + x^2$ are $\pm i$. Hence $\rho = 1$.

(j) The zeroes of $P(x) = 4 - x^2$ are ± 2 . Hence $\rho = 2$.

(k) The zeroes of $P(x) = 3 - x^2$ are $\pm\sqrt{3}$. Hence $\rho = \sqrt{3}$.

(l) The only root of $P(x) = 1 - x$ is 1 . Hence $\rho = 1$.

(m) $p(x) = x/2$ and $q(x) = 3/2$ are analytic for all x .

(n) $p(x) = (1 + x)/2$ and $q(x) = 3/2$ are analytic for all x .

10.(a) If we assume that $y = \sum_{n=0}^{\infty} a_n x^n$, then $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$ and also $y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$. Substituting in the D.E. gives

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=1}^{\infty} n a_n x^n + \alpha^2 \sum_{n=0}^{\infty} a_n x^n = 0.$$

Shifting indices of summation and collecting coefficients of like powers of x yields the equation

$$\left(2 \cdot 1 \cdot a_2 + \alpha^2 a_0\right) x^0 + \left[3 \cdot 2 \cdot a_3 + (\alpha^2 - 1)a_1\right] x^1 + \\ + \sum_{n=2}^{\infty} \left[(n+2)(n+1)a_{n+2} + (\alpha^2 - n^2)a_n\right] x^n = 0.$$

Hence the recurrence relation is $a_{n+2} = (n^2 - \alpha^2)a_n/(n+2)(n+1)$, $n = 0, 1, 2, \ldots$. For the first solution we choose $a_1 = 0$. We find that $a_2 = -\alpha^2 a_0/2 \cdot 1$, $a_3 = 0$, $a_4 = (2^2 - \alpha^2)a_2/4 \cdot 3 = -(2^2 - \alpha^2)\alpha^2 a_0/4!, \ldots$, and then by induction $a_{2m} = -\left[(2m-2)^2 - \alpha^2\right] \ldots (2^2 - \alpha^2)\alpha^2 a_0/(2m)!$, and $a_{2m+1} = 0$, so

$$y_1(x) = 1 - \frac{\alpha^2}{2!}x^2 - \frac{(2^2 - \alpha^2)\alpha^2}{4!}x^4 - \ldots - \frac{\left[(2m-2)^2 - \alpha^2\right] \ldots (2^2 - \alpha^2)\alpha^2}{(2m)!}x^{2m} - \ldots,$$

where we have set $a_0 = 1$. For the second solution we take $a_0 = 0$ and $a_1 = 1$ in the recurrence relation to obtain the desired solution.

(b) If α is an even integer $2k$, then $(2m-2)^2 - \alpha^2 = 4(m-1)^2 - 4k^2$. Thus when $m = k + 1$ all terms in the series for $y_1(x)$ are zero after the x^{2k} term. A similar argument shows that if $\alpha = 2k + 1$, then all terms in $y_2(x)$ are zero after the x^{2k+1} term.

(c) Using the previous parts, we obtain that $p_0(x) = 1$, $p_1(x) = x$, $p_2(x) = 1 - 2x^2$ and $p_3(x) = x - 4x^3/3$.

11. The Taylor series about $x = 0$ for $\sin x$ is $\sin x = x - x^3/3! + x^5/5! - \ldots$. Assuming that $y = \sum_{n=2}^{\infty} a_n x^n$, we find

$$y'' + (\sin x)y = 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + 30a_6 x^4 + 42a_7 x^5 + \ldots +$$

$$+(x - x^3/3! + x^5/5! - \ldots)(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots) =$$

$$= 2a_2 + (6a_3 + a_0)x + (12a_4 + a_1)x^2 + (20a_5 + a_2 - a_0/6)x^3 +$$

$$+(30a_6 + a_3 - a_1/6)x^4 + (42a_7 + a_4 - a_2/3! + a_0/5!)x^5 + \ldots = 0.$$

Hence $a_2 = 0$, $a_3 = -a_0/6$, $a_4 = -a_1/12$, $a_5 = a_0/120$, $a_6 = (a_1 + a_0)/180$, $a_7 = -a_0/7! + a_1/504, \ldots$. We set $a_0 = 1$ and $a_1 = 0$ and obtain $y_1(x) = (1 - x^3/6 + x^5/120 + x^6/180 + \ldots)$. Next we set $a_0 = 0$ and $a_1 = 1$ and obtain $y_2(x) = (x - x^4/12 + x^6/180 + x^7/504 + \ldots)$. Since $p(x) = 1$ and $q(x) = \sin x$, both have $\rho = \infty$; the solution in this case converges for all x, that is, $\rho = \infty$.

18. We know that $e^x = 1 + x + x^2/2! + x^3/3! + \ldots$, and therefore $e^{x^2} = 1 + x^2 + x^4/2! + x^6/3! + \ldots$. Hence if $y = \sum_{n=0}^\infty a_nx^n$, we have $y' = \sum_{n=1}^\infty na_nx^{n-1}$, so

$$a_1 + 2a_2x + 3a_3x^2 + \ldots = (1 + x^2 + x^4/2 + \ldots)(a_0 + a_1x + a_2x^2 + \ldots) =$$

$$= a_0 + a_1x + (a_0 + a_2)x^2 + \ldots.$$

Thus, $a_1 = a_0$, $2a_2 = a_1$, and $3a_3 = a_0 + a_2$, which yield the desired solution.

20. Substituting $y = \sum_{n=0}^\infty a_nx^n$ into the differential equation we obtain that $\sum_{n=1}^\infty na_nx^{n-1} - \sum_{n=0}^\infty a_nx^n = x^2$. Shifting the indices then gives the equation $\sum_{n=0}^\infty [(n+1)a_{n+1} - a_n]\, x^n = x^2$. Equating coefficients of both sides then gives: $a_1 - a_0 = 0, 2a_2 - a_1 = 0, 3a_3 - a_2 = 1$ and $(n+1)a_{n+1} = a_n$ for $n = 3, 4, \ldots$. Thus $a_1 = a_0$, $a_2 = a_1/2 = a_0/2$, $a_3 = 1/3 + a_2/3 = 1/3 + a_0/2 \cdot 3$, $a_4 = a_3/4 = 1/3 \cdot 4 + a_0/2 \cdot 3 \cdot 4 = 2/4! + a_0/4!$, and in general $a_n = a_{n-1}/n = 2/n! + a_0/n!$. Hence

$$y(x) = a_0\left(1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!}\ldots\right) + 2\left(\frac{x^3}{3!} + \frac{x^4}{4!} + \ldots + \frac{x^n}{n!} + \ldots\right).$$

Using the power series for e^x, the first and second sums can be rewritten as $a_0e^x + 2(e^x - 1 - x - x^2/2)$, which is the same solution as we found using methods of Chap. 2.

22. Substituting $y = \sum_{n=0}^\infty a_nx^n$ into the Legendre equation, shifting indices, and collecting coefficients of like powers of x yields

$$[2 \cdot 1 \cdot a_2 + \alpha(\alpha+1)a_0]x^0 + (3 \cdot 2 \cdot a_3 - [2 \cdot 1 - \alpha(\alpha+1)]a_1)x^1 +$$

$$+\sum_{n=2}^{\infty}((n+2)(n+1)a_{n+2} - [n(n+1) - \alpha(\alpha+1)]a_n)x^n = 0.$$

Thus $a_2 = -\alpha(\alpha+1)a_0/2!$, $a_3 = [2 \cdot 1 - \alpha(\alpha+1)]a_1/3! = -(\alpha-1)(\alpha+2)a_1/3!$ and the recurrence relation gives us $(n+2)(n+1)a_{n+2} = -[\alpha(\alpha+1) - n(n+1)]a_n = -(\alpha-n)(\alpha+n+1)a_n$, $n = 2, 3, \ldots$. Setting $a_1 = 0$, $a_0 = 1$ yields a solution with $a_3 = a_5 = a_7 = \ldots = 0$ and $a_4 = \alpha(\alpha-2)(\alpha+1)(\alpha+3)/4\ldots$, and generally, $a_{2m} = (-1)^m[\alpha(\alpha-2)\ldots(\alpha-2m+2)][(\alpha+1)\ldots(\alpha+2m-1)]/(2m)!$. The second linearly independent solution is obtained by setting $a_0 = 0$ and $a_1 = 1$. The coefficients are then $a_2 = a_4 = a_6 = \ldots = 0$ and $a_3 = (\alpha-1)(\alpha+2)/3!$, and $a_5 = -(\alpha-3)(\alpha+4)a_3/5 \cdot 4 = (\alpha-1)(\alpha-3)(\alpha+2)(\alpha+4)/5!$.

26. Using the chain rule, we have

$$\frac{dF(\phi)}{d\phi} = \frac{dF[\phi(x)]}{dx}\frac{dx}{d\phi} = -f'(x)\sin\phi(x) = -f'(x)\sqrt{1-x^2},$$
$$\frac{d^2F(\phi)}{d\phi^2} = \frac{d}{dx}\left[-f'(x)\sqrt{1-x^2}\right]\frac{dx}{d\phi} = (1-x^2)f''(x) - xf'(x),$$

which when substituted into the D.E. yields the desired result.

28. Since $[(1-x^2)y']' = (1-x)^2y'' - 2xy'$, the Legendre Equation, from Problem 22, can be written as shown. Thus, carrying out the multiplications indicated yields the two equations

$$P_m\left[(1-x^2)P_n'\right]' = -n(n+1)P_nP_m$$
$$P_n\left[(1-x^2)P_m'\right]' = -m(m+1)P_nP_m.$$

As long as $n \neq m$, the second equation can be subtracted from the first and the result integrated from -1 to 1 to obtain

$$\int_{-1}^{1}\left\{P_m\left[(1-x^2)P_n'\right]' - P_n\left[(1-x^2)P_m'\right]'\right\}dx = [m(m+1)-n(n+1)]\int_{-1}^{1}P_nP_m dx.$$

The left side may be integrated by parts to yield

$$\left[P_m(1-x^2)P_n' - P_n(1-x^2)P_m'\right]_{-1}^{1} + \int_{-1}^{1}\left[P_m'(1-x^2)P_n' - P_n'(1-x^2)P_m'\right]dx,$$

which is zero. Thus $\int_{-1}^{1}P_n(x)P_m(x)dx = 0$ for $n \neq m$.

5.4

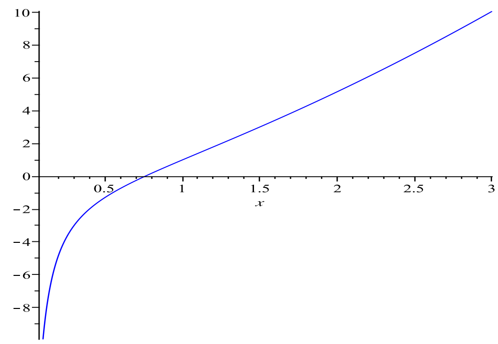
2. This equation is of the form of an Euler equation with x replaced by $x+1$, so we seek solutions of the form $y = (x+1)^r$ for $x+1 > 0$. Substitution of y into the D.E. yields $F(r) = [r(r-1)+3r+3/4](x+1)^r = 0$. Thus $r^2+2r+3/4 = 0$, which gives $r = -3/2, -1/2$. The general solution of the differential equation is then $y = c_1|x+1|^{-1/2} + c_2|x+1|^{-3/2}$, $x \neq -1$.

4. If $y = x^r$, then $F(r) = r(r-1)+3r+5 = 0$. So $r^2+2r+5 = 0$ and then we obtain $r = \left(-2 \pm \sqrt{4-20}\right)/2 = -1 \pm 2i$. Thus the general solution of the differential equation is $y = c_1x^{-1}\cos(2\ln|x|) + c_2x^{-1}\sin(2\ln|x|)$, $x \neq 0$.

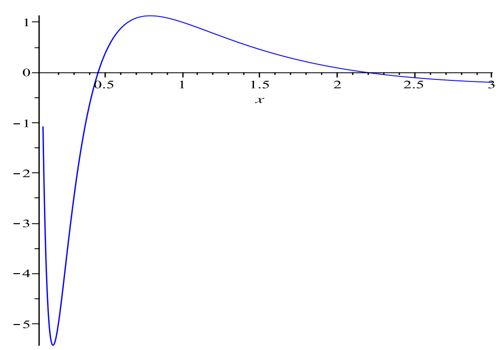
9. Again let $y = x^r$ to obtain $F(r) = r(r-1) - 5r + 9 = 0$, or $(r-3)^2 = 0$. Thus the roots are $x = 3$, 3 and $y = c_1x^3 + c_2x^3\ln|x|$, $x \neq 0$, is the solution of the differential equation.

13. In this case $F(r) = 2r(r-1)+r-3 = 2r^2-r-3 = (2r-3)(r+1) = 0$, so $y = c_1x^{3/2} + c_2x^{-1}$ (since $x_0 = 1$, we do not need $|x|$) and $y' = (3/2)c_1x^{1/2} - c_2x^{-2}$. Setting $x = 1$ in y and y', we obtain $c_1 + c_2 = 1$ and $(3/2)c_1 - c_2 = 4$, which yield

$c_1 = 2$ and $c_2 = -1$. Hence $y = 2x^{3/2} - x^{-1}$. As $x \to 0^+$, we have $y \to -\infty$ due to the second term.



16. We have $F(r) = r(r-1) + 3r + 5 = r^2 + 2r + 5 = 0$. Thus $r_1, r_2 = -1 \pm 2i$ and $y = x^{-1}[c_1 \cos(2\ln x) + c_2 \sin(2\ln x)]$. Then $y(1) = c_1 = 1$ and we obtain that $y' = -x^{-2}[\cos(2\ln x) + c_2 \sin(2\ln x)] + x^{-1}[-\sin(2\ln x)2/x + c_2 \cos(2\ln x)2/x]$, so that $y'(1) = -1 + 2c_2 = -1$, or $c_2 = 0$. Hence $y = x^{-1}\cos(2\ln x)$ for $x > 0$. As $x \to 0^+$ this will oscillate rapidly, with large amplitudes. (The graphing program can't show this, but the form of $y(x)$ clearly indicates this behavior.)



17. Since the coefficients of y, y', and y'' have no common factors, and since $P(x)$ vanishes only at $x = 0$, we conclude that $x = 0$ is a singular point. Writing the differential equation in the form $y'' + p(x)y' + q(x)y = 0$, we get $p(x) = (1-x)/x$ and $q(x) = 1$. Thus for the singular point we have $\lim_{x\to 0} xp(x) = \lim_{x\to 0} 1 - x = 1$, and $\lim_{x\to 0} x^2 q(x) = 0$, thus $x = 0$ is a regular singular point.

21. Writing the differential equation in the form $y'' + p(x)y' + q(x)y = 0$, we find $p(x) = x/(1-x)(1+x)^2$ and $q(x) = (1+x)/(1-x^2)^2$. Therefore $x = \pm 1$ are singular points. Since $\lim_{x\to 1}(x-1)p(x)$ and $\lim_{x\to 1}(x-1)^2 q(x)$ both exist, we conclude that $x = 1$ is a regular singular point. Finally, since $\lim_{x\to -1}(x+1)p(x)$ does not exist, we conclude that $x = -1$ is an irregular singular point.

28. Writing the differential equation in the form $y'' + p(x)y' + q(x)y = 0$, we see that $p(x) = e^x/x$ and $q(x) = (3\cos x)/x$. Thus $x = 0$ is a singular point. Since $xp(x) = e^x$ is analytic at $x = 0$ and $x^2 q(x) = 3x\cos x$ is analytic at $x = 0$, the point

$x = 0$ is a regular singular point.

33. Writing the differential equation in the form $y'' + p(x)y' + q(x)y = 0$, we see that $p(x) = x/\sin x$ and $q(x) = 4/\sin x$. Since $\lim_{x\to 0} q(x)$ does not exist, the point $x_0 = 0$ is a singular point and since neither $\lim_{x\to\pm n\pi} p(x)$ nor $\lim_{x\to\pm n\pi} q(x)$ exist, either, the points $x_0 = \pm n\pi$ are also singular points. To determine whether the singular points are regular or irregular we must use Eq. (31) and the result #7 of multiplication and division of power series from Section 5.1. For $x_0 = 0$, we have

$$xp(x) = \frac{x^2}{\sin x} = \frac{x^2}{x - \frac{x^3}{6} + \dots} = x\left[1 + \frac{x^2}{6} + \dots\right] = x + \frac{x^3}{6} + \dots,$$

which converges about $x_0 = 0$ and thus $xp(x)$ is analytic at $x_0 = 0$. $x^2q(x)$, by similar steps, is also analytic at $x_0 = 0$ and thus $x_0 = 0$ is a regular singular point. For $x_0 = n\pi$,we have

$$(x - n\pi)p(x) = \frac{(x - n\pi)x}{\sin x} = \frac{(x - n\pi)[(x - n\pi) + n\pi]}{\pm(x - n\pi) \mp \frac{(x-n\pi)^3}{6} \pm \dots}$$

$$= \pm[(x - n\pi) + n\pi]\left[1 + \frac{(x - n\pi)^2}{6} + \dots\right],$$

which converges about $x_0 = n\pi$ and thus $(x - n\pi)p(x)$ is analytic at $x = n\pi$. Similarly $(x + n\pi)p(x)$ and $(x \pm n\pi)^2q(x)$ are analytic and thus $x_0 = \pm n\pi$ are regular singular points.

35. Substituting $y = x^r$, we find that $r(r - 1) + \alpha r + 5/2 = 0$ or $r^2 + (\alpha - 1)r + 5/2 = 0$. Thus $r_1, r_2 = \left[-(\alpha - 1) \pm \sqrt{(\alpha - 1)^2 - 10}\right]/2$. In order for solutions to approach zero as $x \to 0$, it is necessary that the real parts of r_1 and r_2 be positive. Suppose that $\alpha > 1$, then $\sqrt{(\alpha - 1)^2 - 10}$ is either imaginary or real and less than $\alpha - 1$; hence the real parts of r_1 and r_2 will be negative. Suppose that $\alpha = 1$, then $r_1, r_2 = \pm i\sqrt{10}$ and the solutions are oscillatory. Suppose that $\alpha < 1$, then $\sqrt{(\alpha - 1)^2 - 10}$ is either imaginary or real and less than $1 - \alpha$; hence the real parts of r_1 and r_2 will be positive. Thus if $\alpha < 1$, the solutions of the differential equation will approach zero as $x \to 0$.

39. In all cases the roots of $F(r) = 0$ are given by Eq. (6) and the forms of the solution are given in Eqs. (25), (26), and (27).

(a) The real part of the root must be positive so, from Eq. (6), $\alpha < 1$. Also $\beta > 0$, since the $\sqrt{(\alpha - 1)^2 - 4\beta}$ term must be less than $|\alpha - 1|$.

(b) Similarly to part (a), if $\alpha < 1$, then here we need $\beta \geq 0$ (a single zero eigenvalue is allowed in this case) or if $\alpha = 1$, then we need $\beta > 0$.

(c) The real part of the root has to be negative, so $\alpha > 1$, and $\beta > 0$ (a negative β value would give us a positive root, $\beta = 0$ would give us a zero root).

(d) The real part of the root must be negative, so $\alpha > 1$, with $\beta \geq 0$ (for $\beta = 0$ one root is zero, which gives a bounded solution as $x \to \infty$). If $\alpha = 1$, then the roots are $\pm\sqrt{-4\beta}$, so $\beta > 0$ will yield oscillatory solutions as $x \to \infty$, which are bounded.

(e) According to (b) and (d) this happens when $\alpha = 1$ and $\beta > 0$.

40. Assume that $y = v(x)x^{r_1}$. Then we obtain that $y' = v(x)r_1x^{r_1-1} + v'(x)x^{r_1}$ and $y'' = v(x)r_1(r_2-1)x^{r_1-2} + 2v'(x)r_1x^{r_1-1} + v''(x)x^{r_1}$. Substituting in the D.E. and collecting terms yields $x^{r_1+2}v'' + (\alpha + 2r_1)x^{r_1+1}v' + [r_1(r_1-1) + \alpha r_1 + \beta]\,x^{r_1}v = 0$. Now we make use of the fact that r_1 is a double root of $f(r) = r(r-1) + \alpha r + \beta$. This means that $f(r_1) = 0$ and $f'(r_1) = 2r_1 - 1 + \alpha = 0$. Hence the D.E. for v reduces to $x^{r_1+2}v'' + x^{r_1+1}v'$. Since $x > 0$, we may divide by x^{r_1+1} to obtain $xv'' + v' = 0$. Thus $v(x) = \ln x$ and a second solution is $y = x^{r_1}\ln x$.

41. Substituting $y = \sum_{n=0}^{\infty} a_n x^n$ into the differential equation yields

$$2\sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} + 3\sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^{n+1} = 0.$$

The last sum becomes $\sum_{n=2}^{\infty} a_{n-2}x^{n-1}$ (let $m = n+2$ and then replace m by n), the first term of the middle sum is $3a_1$, and thus we have

$$3a_1 + \sum_{n=2}^{\infty}\{[2n(n-1) + 3n]a_n + a_{n-2}\}x^{n-1} = 0.$$

Hence $a_1 = 0$ and $a_n = -a_{n-2}/n(2n+1)$, which is the desired recurrence relation. Thus all even coefficients are found in terms of a_0 and all odd coefficients are zero, thereby yielding only one solution of the desired form. The result is

$$y = a_0\left(1 - \frac{x^2}{2\cdot 5} + \frac{x^4}{2\cdot 4\cdot 5\cdot 9} - \ldots\right).$$

43. If $\xi = 1/x$, then

$$\frac{dy}{dx} = \frac{dy}{d\xi}\frac{d\xi}{dx} = -\frac{1}{x^2}\frac{dy}{d\xi} = -\xi^2\frac{dy}{d\xi},$$

$$\frac{d^2y}{dx^2} = \frac{d}{d\xi}\left(-\xi^2\frac{dy}{d\xi}\right)\frac{d\xi}{dx} = \left(-2\xi\frac{dy}{d\xi} - \xi^2\frac{d^2y}{d\xi^2}\right)\left(-\frac{1}{x^2}\right) =$$

$$\xi^4\frac{d^2y}{d\xi^2} + 2\xi^3\frac{dy}{d\xi}.$$

Substituting in the differential equation, we have

$$P(1/\xi)\left[\xi^4\frac{d^2y}{d\xi^2} + 2\xi^3\frac{dy}{d\xi}\right] + Q(1/\xi)\left[-\xi^2\frac{dy}{d\xi}\right] + R(1/\xi)y = 0,$$

or

$$\xi^4 P(1/\xi)\frac{d^2y}{d\xi^2} + \left[2\xi^3 P(1/\xi) - \xi^2 Q(1/\xi)\right]\frac{dy}{d\xi} + R(1/\xi)y = 0.$$

The result follows from the theory of singular points at $\xi = 0$.

45. Since $p(x) = x^2$, $Q(x) = x$, and $R(x) = -4$, we have

$$f(\xi) = \left[2P(1/\xi)/\xi - Q(1/\xi)/\xi^2\right]/P(1/\xi) = 2/\xi - 1/\xi = 1/\xi$$

and $g(\xi) = R(1/\xi)/\xi^4 P(1/\xi) = -4/\xi^2$. Thus the point at infinity is a singular point. Since both $\xi f(\xi)$ and $\xi^2 g(\xi)$ are analytic at $\xi = 0$, the point at infinity is a regular singular point.

47. Since $p(x) = x^2$, $Q(x) = x$, and $R(x) = x^2 - \nu^2$, we have

$$f(\xi) = \left[2P(1/\xi)/\xi - Q(1/\xi)/\xi^2\right]/P(1/\xi) = 2/\xi - 1/\xi = 1/\xi$$

and $g(\xi) = R(1/\xi)/\xi^4 P(1/\xi) = (1/\xi^2 - \nu^2)/\xi^2 = 1/\xi^4 - \nu^2/\xi^2$. Thus the point at infinity is a singular point. Although $\xi f(\xi) = 1$ is analytic at $\xi = 0$, $\xi^2 g(\xi) = 1/\xi^2 - \nu^2$ is not, so the point at infinity is an irregular singular point.

5.5

2.(a) If the D.E. is put in the standard form $y'' + p(x)y + q(x)y = 0$, then $p(x) = x^{-1}$ and $q(x) = 1 - 1/9x^2$. Thus $x = 0$ is a singular point. Since $xp(x) \to 1$ and $x^2 q(x) \to -1/9$ as $x \to 0$, it follows that $x = 0$ is a regular singular point.

(b) In determining a series solution of the D.E. it is more convenient to leave the equation in the form given rather then divide by x^2, the coefficient of y''. If we substitute $y = \sum_{n=0}^{\infty} a_n x^{n+r}$, we have

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r} + \left(x^2 - \frac{1}{9}\right)\sum_{n=0}^{\infty} a_n x^{n+r} = 0.$$

Note that

$$x^2 \sum_{n=0}^{\infty} a_n x^{n+r} = \sum_{n=0}^{\infty} a_n x^{n+r+2} = \sum_{n=2}^{\infty} a_{n-2} x^{n+r}.$$

Thus we have

$$\left[r(r-1) + r - \frac{1}{9}\right] a_0 x^r + \left[(r+1)r + a(r+1) - \frac{1}{9}\right] a_1 x^{r+1} +$$

$$+ \sum_{n=2}^{\infty}\left\{\left[(n+r)(n+r-1) + (n+r) - \frac{1}{9}\right] a_n + a_{n-2}\right\} x^{n+r} = 0.$$

From the first term, the indicial equation is $r^2 - 1/9 = 0$ with roots $r_1 = 1/3$ and $r_2 = -1/3$. For either value of r it is necessary to take $a_1 = 0$ in order that the coefficient of x^{r+1} be zero. The recurrence relation is $a_n = -a_{n-2}/[(n+r)^2 - 1/9]$.

(c) For $r = 1/3$ we have

$$a_n = \frac{-a_{n-2}}{(n+1/3)^2 - (1/3)^2} = -\frac{a_{n-2}}{(n+2/3)\,n}, n = 2, 3, 4 \ldots.$$

Since $a_1 = 0$, it follows from the recurrence relation that $a_3 = a_5 = a_7 = \ldots = 0$. For the even coefficients it is convenient to let $n = 2m, m = 1, 2, 3, \ldots$. Then $a_{2m} = -a_{2m-2}/2^2 m\,(m + 1/3)$. The first few coefficients are given by

$$a_2 = \frac{(-1)a_0}{2^2\,(1+1/3)\,1}, a_4 = \frac{(-1)a_2}{2^2\,(2+1/3)\,2} = \frac{a_0}{2^4\,(1+1/3)\,(2+1/3)\,2!}$$
$$a_6 = \frac{(-1)a_4}{2^2\,(3+1/3)\,3} = \frac{(-1)a_0}{2^6\,(1+1/3)\,(2+1/3)\,(3+1/3)\,3!},$$

and the coefficient of x^{2m} for $m = 1, 2, \ldots$ is

$$a_{2m} = \frac{(-1)^m a_0}{2^{2m} m!\,(1+1/3)\,(2+1/3)\ldots(m+1/3)}.$$

Thus one solution (on setting $a_0 = 1$) is

$$y_1(x) = x^{1/3}\left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{m!\,(1+1/3)\,(2+1/3)\ldots(m+1/3)}\left(\frac{x}{2}\right)^{2m}\right]$$

(d) Since $r_2 = -1/3 \neq r_1$ and $r_1 - r_2 = 2/3$ is not an integer, we can calculate a second series solution corresponding to $r = -1/3$. The recurrence relation is $n(n - 2/3)a_n = -a_{n-2}$, which yields the desired solution following the steps in part (c). Note that $a_1 = 0$, as in the first solution, and thus all the odd coefficients are zero.

4.(a) Putting the D.E. in the form $y'' + p(x)y' + q(x)y = 0$, we see that $p(x) = 1/x$ and $q(x) = -1/x$. Thus $x = 0$ is a singular point, and since $xp(x) \to 1$ and $x^2 q(x) \to 0$, as $x \to 0$, $x = 0$ is a regular singular point.

(b) Substituting $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ in $xy'' + y' - y = 0$ and shifting indices, we obtain

$$\sum_{n=-1}^{\infty} a_{n+1}(r+n+1)(r+n)x^{n+r} + \sum_{n=-1}^{\infty} a_{n+1}(r+n+1)x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0,$$

or

$$[r(r-1)+r]a_0 x^{-1+r} + \sum_{n=0}^{\infty}\left[(r+n+1)^2 a_{n+1} - a_n\right]x^{n+r} = 0.$$

From the first coefficient we find $r^2 = 0$ is the indicial equation, and from the coefficient of x^{n+r} we find the recurrence relation is $a_{n+1} = a_n/(n+1+r)^2$.

(c) Setting $r = 0$ in the reccurence relation, we find $(n+1)^2 a_{n+1} = a_n, n = 0, 1, 2, \ldots$. The coefficients are $a_1 = a_0$, $a_2 = a_1/2^2 = a_0/2^2$, $a_3 = a_2/3^2 = a_0/3^2 \cdot 2^2$, $a_4 = a_3/4^2 = a_0/4^2 \cdot 3^2 \cdot 2^2, \ldots$ and $a_n = a_0/(n!)^2$. Thus one solution (on setting $a_0 = 1$) is $y = \sum_{n=0}^{\infty} x^n/(n!)^2$.

(d) Since the indicial equation has only one root, we only have one solution of the form $y = x^r \sum_{n=0}^{\infty} a_n x^{n+r}$.

11. (a) If we make the change of variable $t = x - 1$ and let $y = u(t)$, then the Legendre equation transforms to $(t^2 + 2t)u''(t) + 2(t+1)u'(t) - \alpha(\alpha+1)u(t) = 0$. Since $x = 1$ is a regular singular point of the original equation, we know that $t = 0$ is a regular singular point of the transformed equation. Substituting $u = \sum_{n=0}^{\infty} a_n t^{n+r}$ in the transformed equation and shifting indices, we obtain

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n t^{n+r} + 2\sum_{n=-1}^{\infty}(n+r+1)(n+r)a_{n+1}t^{n+r} +$$

$$+2\sum_{n=0}^{\infty}(n+r)a_n t^{n+r} + 2\sum_{n=-1}^{\infty}(n+r+1)a_{n+1}t^{n+r} - \alpha(\alpha+1)\sum_{n=0}^{\infty} a_n t^{n+r} = 0,$$

or

$$[2r(r-1) + 2r]a_0 t^{r-1} +$$

$$+\sum_{n=0}^{\infty}\{2(n+r+1)^2 a_{n+1} + [(n+r)(n+r+1) - \alpha(\alpha+1)]a_n\}t^{n+r} = 0.$$

The indicial equation is $2r^2 = 0$, so $r = 0$ is a double root. Thus there will be only one series solution of the form $y = \sum_{n=0}^{\infty} a_n t^{n+r}$.

(b) The recurrence relation is

$$2(n+1)^2 a_{n+1} = [\alpha(\alpha+1) - n(n+1)]a_n, n = 0, 1, 2, \ldots.$$

We have

$$a_1 = [\alpha(\alpha+1)]a_0/2\cdot 1^2, a_2 = [\alpha(\alpha+1)][\alpha(\alpha+1) - 1\cdot 2]a_0/2^2\cdot 2^2\cdot 1^2,$$

$$a_3 = [\alpha(\alpha+1)][\alpha(\alpha+1) - 1\cdot 2][\alpha(\alpha+1) - 2\cdot 3]a_0/2^3\cdot 3^2\cdot 2^2\cdot 1^2, \ldots, \text{ and}$$

$$a_n = [\alpha(\alpha+1)][\alpha(\alpha+1) - 1\cdot 2]\ldots[\alpha(\alpha+1) - (n-1)\cdot n]a_0/2^n(n!)^2.$$

Reverting to the variable x, it follows that one solution of the Legendre equation in powers of $x - 1$ is

$$y_1(x) = \sum_{n=0}^{\infty}[\alpha(\alpha+1)][\alpha(\alpha+1) - 1\cdot 2]\ldots[\alpha(\alpha+1) - (n-1)\cdot n](x-1)^n/2^n(n!)^2$$

where we have set $a_0 = 1$, which is equivalent to the answer in the text if a (-1) is factored out of each square bracket.

14. (a) The standard form is $y'' + p(x)y' + q(x)y = 0$, with $p(x) = 1/x$ and $q(x) = 1$. Thus $x = 0$ is a singular point; and since $xp(x) \to 1$ and $x^2q(x) \to 0$ as $x \to 0$, $x = 0$ is a regular singular point.

(b) Substituting $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ into $x^2y'' + xy' + x^2y = 0$ and shifting indices appropriately, we obtain

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r} + \sum_{n=2}^{\infty} a_{n-2}x^{n+r} = 0,$$

or

$$[r(r-1)+r]a_0x^r + [(1+r)r+1+r]a_1x^{r+1} + \sum_{n=2}^{\infty}\left[(n+r)^2a_n + a_{n-2}\right]x^{n+r} = 0.$$

The indicial equation is, $r^2 = 0$ so $r = 0$ is a double root. It is necessary to take $a_1 = 0$ in order that the coefficient of x^{r+1} be zero.

(c) The recurrence relation is $n^2a_n = -a_{n-2}$, $n = 2, 3, \ldots$. Since $a_1 = 0$ it follows that $a_3 = a_5 = a_7 = \ldots = 0$. For the even coefficients we let $n = 2m$, $m = 1, 2, \ldots$. Then $a_{2m} = -a_{2m-2}/2^2m^2$ so $a_2 = -a_0/2^2 \cdot 1^2$, $a_4 = a_0/2^2 \cdot 2^2 \cdot 1^2 \cdot 2^2, \ldots$, so we get $a_{2m} = (-1)^m a_0/2^{2m}(m!)^2$. Thus one solution of the Bessel equation of order zero is $J_0(x) = 1 + \sum_{m=1}^{\infty}(-1)^m x^{2m}/2^{2m}(m!)^2$ where we have set $a_0 = 1$.

(d) Using the ratio test, it can be shown that the series converges for all x. Also note that $J_0(x) \to 1$ as $x \to 0$.

15. In order to determine the form of the integral for x near zero, we must study the integrand for x small. Using the above series for J_0, we have

$$\frac{1}{x[J_0(x)]^2} = \frac{1}{x[1-x^2/2+\ldots]^2} = \frac{1}{x[1-x^2+\ldots]} = \frac{1}{x}[1+x^2+\ldots]$$

for x small. Thus

$$y_2(x) = J_0(x)\int\frac{dx}{x[J_0(x)]^2} = J_0(x)\int\left[\frac{1}{x}+x+\ldots\right]dx = J_0(x)\left[\ln x + \frac{x^2}{x}+\ldots\right],$$

and it is clear that $y_2(x)$ will contain a logarithmic term.

16.(a) Putting the D.E. in the standard form $y'' + p(x)y' + q(x)y = 0$, we see that $p(x) = 1/x$ and $q(x) = (x^2-1)/x^2$. Thus $x = 0$ is a singular point and since $xp(x) \to 1$ and $x^2q(x) \to -1$ as $x \to 0$, $x = 0$ is a regular singular point.

(b) Substituting $y = \sum_{n=0}^{\infty} a_nx^{n+r}$ into $x^2y'' + xy' + (x^2-1)y = 0$, shifting indices appropriately and collecting coefficients of common powers of x, we obtain

$$[r(r-1)+r-1]a_0x^r + [(1+r)r+1+r-1]a_1x^{r+1} + \sum_{n=2}^{\infty}\left\{\left[(n+r)^2-1\right]a_n + a_{n-2}\right\}x^{n+r} = 0.$$

The indicial equation is $r^2 - 1 = 0$ so the roots are $r_1 = 1$ and $r_2 = -1$.

(c) For either value of r it is necessary to take $a_1 = 0$ in order that the coefficient of x^{r+1} be zero. The recurrence relation is $[(n+r)^2-1]a_n = -a_{n-2}, n = 2, 3, 4, \ldots$. For $r = 1$ we have $a_n = -a_{n-2}/[n(n+2)], n = 2, 3, 4, \ldots$. Since $a_1 = 0$ it follows that $a_3 = a_5 = a_7 = \ldots = 0$. Let $n = 2m$. Then $a_{2m} = -a_{2m-2}/2^2m(m+1)$, $m = 1, 2, \ldots$, so $a_2 = -a_0/2^2 \cdot 1 \cdot 2$, $a_4 = -a_2/2^2 \cdot 1 \cdot 2 \cdot 3 = a_0/2^2 \cdot 2^2 \cdot 1 \cdot 2 \cdot 2 \cdot$

$3, \ldots$, and $a^{2m} = (-1)^m a_0/2^{2m} m!(m+1)!$. Thus one solution (set $a_0 = 1/2$) of the Bessel equation of order 1 is

$$J_1(x) = \frac{x}{2}\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(n+1)!n!2^{2n}}.$$

(d) The ratio test shows that the series converges for all x. Also note that $J_1(x) \to 0$ as $x \to 0$.

(e) For $r = -1$ the recurrence relation is $[(n-1)^2 - 1]a_n = -a_{n-2}, n = 2, 3, \ldots$, so for $n = 2$ the coefficient of a_2 is zero and we cannot calculate a_2. Consequently, it is not possible to find a series solution of the form $x^{-1}\sum_{n=0}^{\infty} b_n x^n$.

5.6

1.(a) The differential equation has the form $P(x)y'' + Q(x)y' + R(x)y = 0$ with $P(x) = x$, $Q(x) = 2x$, and $R(x) = 6e^x$. From this we find $p(x) = Q(x)/P(x) = 2$ and $q(x) = R(x)/P(x) = 6e^x/x$, and thus $x = 0$ is a singular point. Since $xp(x) = 2x$ and $x^2q(x) = 6xe^x$ are analytic at $x = 0$, we conclude that $x = 0$ is a regular singular point.

(b) We have $xp(x) \to 0 = p_0$ and $x^2q(x) \to 0 = q_0$ as $x \to 0$ and thus Eq. (7), the indicial equation, is $F(r) = r(r-1) = 0$ which has roots $r_1 = 1$ and $r_2 = 0$. These are the exponents of the singularity at $x = 0$.

3.(a) The differential equation has the form $P(x)y'' + Q(x)y' + R(x)y = 0$ with $P(x) = x(x-1)$, $Q(x) = 6x^2$, and $R(x) = 3$. Since $P(x)$, $Q(x)$, and $R(x)$ are polynomials with no common factors and $P(0) = 0$ and $P(1) = 0$, we conclude that $x = 0$ and $x = 1$ are singular points. The first point, $x = 0$, can be shown to be a regular singular point using steps similar to those shown in Problem 1. The indicial equation is $F(r) = r(r-1)$ and the exponents are 1 and 0 here. For $x = 1$, we must put the differential equation in the form seen in Example (1). To do this, divide the differential equation by x and multiply by $(x-1)$ to obtain $(x-1)^2y'' + 6x(x-1)y + 3(x-1)y/x = 0$. Comparing this to Example (1), we find that $(x-1)p(x) = 6x$ and $(x-1)^2q(x) = 3(x-1)/x$, which are both analytic at $x = 1$, and hence $x = 1$ is a regular singular point.

(b) These last two expressions are $p_0 = 6$ and $q_0 = 0$, respectively, at $x = 1$, and thus the indicial equation is $F(r) = r(r-1) + 6r + 0 = r(r+5) = 0$. The exponents are 0 and -5 at this point.

9.(a) For this differential equation,

$$p(x) = \frac{-(1+x)}{x^2(1-x)} \qquad \text{and} \qquad q(x) = \frac{2}{x(1-x)},$$

and thus $x = 0, 1$ are singular points. Since $xp(x)$ is not analytic at $x = 0$, $x = 0$ is not a regular singular point. Looking at

$$(x-1)p(x) = \frac{1+x}{x^2} \qquad \text{and} \qquad (x-1)^2 q(x) = \frac{2(1-x)}{x},$$

we see that $x = 1$ is a regular singular point.

(b) As in Example (1),

$$p_0 = \lim_{x\to 1}(x-1)p(x) = 2 \qquad \text{and} \qquad q_0 = \lim_{x\to 1}(x-1)^2 q(x) = 0.$$

Thus the indicial equation is $F(r) = r^2 + r$ and $r_1 = 0$ and $r_2 = -1$.

13.(a) Note that $p(x) = 1/x$ and $q(x) = -1/x$. Furthermore, $xp(x) = 1$ and $x^2 q(x) = -x$. It follows that

$$p_0 = \lim_{x\to 0}(1) = 1 \quad \text{and} \quad q_0 = \lim_{x\to 0}(-x) = 0,$$

and therefore $x = 0$ is a regular singular point.

(b) The indicial equation is given by $r(r-1) + r = 0$, that is, $r^2 = 0$, with roots $r_1 = r_2 = 0$. This implies we can use Eq.(18) for the second solution.

(c) Let $y = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$. Substitution into the ODE results in

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\,x^{n+1} + \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n - \sum_{n=0}^{\infty}a_n x^n = 0\,.$$

After adjusting the indices in the first series, we obtain

$$a_1 - a_0 + \sum_{n=1}^{\infty}\left[n(n+1)a_{n+1} + (n+1)a_{n+1} - a_n\right]x^n = 0\,.$$

Setting the coefficients equal to zero, it follows that for $n \geq 0$,

$$a_{n+1} = \frac{a_n}{(n+1)^2}\,.$$

So for $n \geq 1$,

$$a_n = \frac{a_{n-1}}{n^2} = \frac{a_{n-2}}{n^2(n-1)^2} = \ldots = \frac{1}{(n!)^2}\,a_0\,.$$

With $a_0 = 1$, one solution is

$$y_1(x) = 1 + x + \frac{1}{4}x^2 + \frac{1}{36}x^3 + \ldots + \frac{1}{(n!)^2}\,x^n + \ldots\,.$$

For a second solution, set $y_2(x) = y_1(x)\,\ln\,x\, +\, b_1 x + b_2 x^2 + \ldots + b_n x^n + \ldots$. Substituting into the ODE, we obtain

$$L\left[y_1(x)\right]\cdot \ln\,x + 2\,y_1'(x) + L\left[\sum_{n=1}^{\infty} b_n\,x^n\right] = 0\,.$$

Since $L\,[y_1(x)] = 0$, it follows that

$$L\left[\sum_{n=1}^{\infty} b_n\, x^n\right] = -2\, y_1'(x)\,.$$

More specifically,

$$b_1 + \sum_{n=1}^{\infty} \left[n(n+1)b_{n+1} + (n+1)b_{n+1} - b_n\right] x^n = -2 - x - \frac{1}{6}x^2 - \frac{1}{72}x^3 - \frac{1}{1440}x^4 - \ldots$$

Equating the coefficients, we obtain the system of equations $b_1 = -2$, $4b_2 - b_1 = -1$, $9b_3 - b_2 = -1/6$, $16b_4 - b_3 = -1/72, \ldots$. Solving these equations for the coefficients, $b_1 = -2$, $b_2 = -3/4$, $b_3 = -11/108, b_4 = -25/3456\,, \ldots$. Therefore a second solution is

$$y_2(x) = y_1(x)\,\ln\,x + \left[-2x - \frac{3}{4}x^2 - \frac{11}{108}x^3 - \frac{25}{3456}x^4 - \ldots\right].$$

17.(a) We have

$$p(x) = \frac{\sin x}{x^2} \qquad \text{and} \qquad q(x) = \frac{\cos x}{x^2},$$

so that $x = 0$ is a singular point. Note that $xp(x) = \sin x/x \to 1 = p_0$ as $x \to 0$ and $x^2 q(x) = -\cos x \to -1 = q_0$ as $x \to 0$. In order to assert that $x = 0$ is a regular singular point we must demonstrate that $xp(x)$ and $x^2 q(x)$, with $xp(x) = 1$ at $x = 0$ and $x^2 q(x) = -1$ at $x = 0$, have convergent power series (are analytic) about $x = 0$. We know that $\cos x$ is analytic, so we need only consider $\sin x/x$. Now

$$\sin x = \sum_{n=0}^{\infty} (-1)^n x^{2n+1}/(2n+1)!$$

for $-\infty < x < \infty$, so

$$\sin x/x = \sum_{n=0}^{\infty} (-1)^n x^{2n}/(2n+1)!$$

and hence is analytic. Thus we conclude that $x = 0$ is a regular singular point.

(b) From part (a) it follows that the indicial equation is $r(r-1) + r - 1 = r^2 - 1 = 0$ and the roots are $r_1 = 1$, $r_2 = -1$.

(c) To find the first few terms of the solution corresponding to $r_1 = 1$, assume that

$$y(x) = x\sum_{n=0}^{\infty} a_n x^n = x(a_0 + a_1 x + a_2 x^2 + \ldots) = a_0 x + a_1 x^2 + a_2 x^3 + \ldots.$$

Substituting this series for y in the differential equation and expanding $\sin x$ and $\cos x$ about $x = 0$ yields

$$\begin{gathered} x^2(2a_1 + 6a_2 x + 12a_3 x^2 + 20a_4 x^3 + \ldots) + \\ +(x - x^3/3! + x^5/5! - \ldots)(a_0 + 2a_1 x + 3a_2 x^2 + 4a_3 x^3 + 5a_4 x^4 + \ldots) \\ -(1 - x^2/2! + x^4/4! - \ldots)(a_0 x + a_1 x^2 + a_2 x^3 + a_3 x^4 + a_4 x^5 + \ldots) = 0. \end{gathered}$$

Collecting terms, we have

$$(a_0 - a_0)x + (2a_1 + 2a_1 - a_1)x^2 + (6a_2 + 3a_2 - a_0/6 - a_2 + a_0/2)x^3 +$$

$$+(12a_3 + 4a_3 - 2a_1/6 - a_3 + a_1/2)x^4 +$$

$$+(20a_4 + 5a_4 - 3a_2/6 + a_0/120 - a_4 + a_2/2 - a_0/24)x^5 + \ldots = 0.$$

Simplifying yields $3a_1x^2 + (8a_2 + a_0/3)x^3 + (15a_3 + a_1/6)x^4 + (24a_4 - a_0/30)x^5 + \ldots = 0$. Thus, $a_1 = 0,\ a_2 = -a_0/4!,\ a_3 = 0,\ a_4 = a_0/6!, \ldots$. Hence $y_1(x) = x - x^3/4! + x^5/6! + \ldots$ where we have set $a_0 = 1$. For the second solution we use a variation of Equation (24) similar to Equation (18):

$$y_2(x) = ay_1(x)\ln x + x^{-1}\left(1 + \sum_{n=1}^{\infty} c_n x^n\right) =$$

$$ay_1(x)\ln x + \frac{1}{x} + c_1 + c_2x + c_3x^2 + c_4x^3 + \ldots,$$

so we obtain that $y_2' = ay_1'\ln x + ay_1x^{-1} - x^{-2} + c_2 + 2c_3x + 3c_4x^2 + \ldots$, and $y_2'' = ay_1''\ln x + 2ay_1'x^{-1} - ay_1x^{-2} + 2x^{-3} + 2c_3 + 3c_4x + \ldots$. When these are substituted in the given differential equation the terms including $\ln x$ will appear as $a[x^2y_1'' + \sin xy_1' - \cos xy_1]$, which is zero since y_1 is a solution. For the remainder of terms, use $y_1 = x - x^3/24 + x^5/720$ and the $\cos x$ and $\sin x$ series as shown earlier to obtain $-c_1 + (2/3 + 2a)x + (3c_3 + c_1/2)x^2 + (4/45 + c_2/3 + 8c_4)x^3 + \ldots = 0$. These yield $c_1 = 0$, $a = -1/3$, $c_3 = 0$, and $c_4 = -c_2/24 - 1/90$. We may take $c_2 = 0$, since this term will simply generate $y_1(x)$ over again. Thus

$$y_2(x) = -\frac{1}{3}y_1(x)\ln x + x^{-1} - \frac{1}{90}x^3.$$

If a computer algebra system is used, then additional terms in each series may be obtained without much additional effort. The next terms, in each case, are shown here:

$$y_1(x) = x - \frac{x^3}{24} + \frac{x^5}{720} - \frac{43x^7}{1451520} + \ldots \qquad \text{and}$$

$$y_2(x) = -\frac{1}{3}y_1(x)\ln x + \frac{1}{x}\left[1 - \frac{x^4}{90} + \frac{41x^6}{120960} - \ldots\right].$$

18.(a) We first write the D.E. in the standard form as given for Theorem 5.6.1 except that we are expanding in powers of $(x-1)$ rather than powers of x:

$$(x-1)^2y'' + (x-1)[(x-1)/2\ln x]y' + [(x-1)^2/\ln x]y = 0.$$

Since $\ln 1 = 0$, $x = 1$ is a singular point. To show it is a regular singular point of this D.E. we must show that $(x-1)/\ln x$ is analytic at $x = 1$; it will then follow that $(x-1)^2/\ln x = (x-1)[(x-1)/\ln x]$ is also analytic at $x = 1$. If we expand $\ln x$ in a Taylor series about $x = 1$, we find that

$$\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \ldots.$$

Thus

$$(x-1)/\ln x = \left[1 - \frac{1}{2}(x-1) + \frac{1}{3}(x-1)^2 - \dots\right]^{-1} = 1 + \frac{1}{2}(x-1) + \dots$$

has a power series expansion about $x = 1$, and hence is analytic.

(b) We can use the above result to obtain the indicial equation at $x = 1$. We have

$$(x-1)^2 y'' + (x-1)\left[\frac{1}{2} + \frac{1}{4}(x-1) + \dots\right] y' + \left[(x-1) + \frac{1}{2}(x-1)^2 + \dots\right] y = 0.$$

Thus $p_0 = 1/2$, $q_0 = 0$, and the indicial equation is $r(r-1) + r/2 = 0$. Hence $r = 1/2$ and $r = 0$.

(c) In order to find the first three nonzero terms in a series solution corresponding to $r = 1/2$, it is better to keep the differential equation in its original form and to substitute the above power series for $\ln x$:

$$\left[(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \dots\right] y'' + \frac{1}{2}y' + y = 0.$$

Next we substitute $y = a_0(x-1)^{1/2} + a_1(x-1)^{3/2} + a_2(x-1)^{5/2} + \dots$ and collect coefficients of like powers of $(x-1)$, which are then set equal to zero. This requires some algebra before we find that $6a_1/4 + 9a_0/8 = 0$ and $5a_2 + 5a_1/8 - a_0/12 = 0$. These equations yield $a_1 = -3a_0/4$ and $a_2 = 53a_0/480$. With $a_0 = 1$ we obtain the solution

$$y_1(x) = (x-1)^{1/2} - \frac{3}{4}(x-1)^{3/2} + \frac{53}{480}(x-1)^{5/2} + \dots.$$

(d) Since the radius of convergence of the Taylor series for $(x-1)/\ln x$ is 1, we would expect $\rho = 1$.

20.(a) If we write the D.E. in the standard form as given in Theorem 5.6.1, we obtain $x^2y'' + x[\alpha/x]y' + [\beta/x]y = 0$ where $xp(x) = \alpha/x$ and $x^2g(x) = \beta/x$. Neither of these terms are analytic at $x = 0$, so $x = 0$ is an irregular singular point.

(b) Substituting $y = x^r \sum_{n=0}^{\infty} a_n x^n$ in $x^3y'' + \alpha xy' + \beta y = 0$ gives

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r+1} + \alpha\sum_{n=0}^{\infty}(n+r)a_n x^{n+r} + \beta\sum_{n=0}^{\infty}a_n x^{n+r} = 0.$$

Shifting the index in the first series and collecting coefficients of common powers of x, we obtain

$$(\alpha r + \beta)a_0 x^r + \sum_{n=1}^{\infty}\left\{(n+r-1)(n+r-2)a_{n-1} + [\alpha(n+r) + \beta]a_n\right\} x^{n+r} = 0.$$

Thus the indicial equation is $\alpha r + \beta = 0$ with the single root $r = -\beta/\alpha$.

(c) From part (b), the recurrence relation is

$$a_n = -\frac{(n+r-1)(n+r-2)a_{n-1}}{\alpha(n+r)+\beta} = -\frac{\left(n-\frac{\beta}{\alpha}-1\right)\left(n-\frac{\beta}{\alpha}-2\right)a_{n-1}}{\alpha n},$$

$n = 1, 2, \ldots$ for $r = -\beta/\alpha$. For $\beta/\alpha = -1$, $a_n = -n(n-1)a_{n-1}/\alpha n$, so that $a_1 = 0 \cdot a_0 = 0$. Since all other a_n are multiples of a_1, and hence are zero, $y(x) = x$ is the solution. Similarly for $\beta/\alpha = 0$, $a_n = -(n-1)(n-2)a_{n-1}/\alpha n$ and again for $n = 1$, $a_1 = 0$ and $y(x) = 1$ is the solution. Continuing in this fashion, we see that the series solution will terminate for β/α any positive integer as well as 0 and -1. For other values of β/α, we have

$$\left|\frac{a_n}{a_{n-1}}\right| = \frac{\left(n-\frac{\beta}{\alpha}-1\right)\left(n-\frac{\beta}{\alpha}-2\right)}{\alpha n},$$

which approaches ∞ as $n \to \infty$ and thus the ratio test yields a zero radius of convergence.

21.(a) Note that

$$p(x) = \frac{\alpha}{x^s} \text{ and } q(x) = \frac{\beta}{x^t}\,.$$

It follows that

$$\lim_{x\to 0} x\,p(x) = \lim_{x\to 0} \alpha\, x^{1-s} \quad \text{and} \quad \lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} \beta\, x^{2-t}.$$

Hence if $s > 1$ or $t > 2$, one or both of the limits does not exist. Therefore $x = 0$ is an irregular singular point.

(b) Substituting $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ in the differential equation gives

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} + \alpha\sum_{n=0}^{\infty}(n+r)a_n x^{n+r+1-s} + \beta\sum_{n=0}^{\infty} a_n x^{n+r+2-t} = 0.$$

If $s = 2$ and $t = 2$ the first term in each of the three series is $r(r-1)a_0x^r$, $\alpha r a_0 x^{r-1}$, and $\beta a_0 x^r$, respectively. Thus the indicial equation is $F(r) = \alpha r a_0 = 0$, which requires $r = 0$. Hence there is at most one solution of the assumed form.

(c) Let $y = a_0x^r + a_1x^{r+1} + \ldots + a_nx^{r+n} + \ldots$. Write the ODE as

$$x^3y'' + \alpha\, x^2 y' + \beta\, y = 0\,.$$

Substitution of the assumed solution results in

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r+1} + \alpha\sum_{n=0}^{\infty}(n+r)a_n x^{n+r+1} + \beta\sum_{n=0}^{\infty} a_n x^{n+r} = 0.$$

Adjusting the indices, we obtain

$$\sum_{n=1}^{\infty}(n-1+r)(n+r-2)a_{n-1}x^{n+r} + \alpha\sum_{n=1}^{\infty}(n-1+r)a_{n-1}x^{n+r} +$$

$$+\beta \sum_{n=0}^{\infty} a_n x^{n+r} = 0.$$

Combining the series,

$$\beta\, a_0 x^r + \sum_{n=1}^{\infty} A_n\, x^{n+r} = 0\,,$$

in which $A_n = \beta\, a_n + (n-1+r)(n+r+\alpha-2)a_{n-1}$. Setting the coefficients equal to zero, we have $a_0 = 0$. But for $n \geq 1$,

$$a_n = \frac{(n-1+r)(n+r+\alpha-2)}{\beta}\, a_{n-1}.$$

Therefore, regardless of the value of r, it follows that $a_n = 0$, for $n = 1, 2, \ldots$.

(d) In order for the indicial equation to be quadratic in r it is necessary that the first term in the first series contribute to the indicial equation. This means that the first term in the second and the third series cannot have powers less than x^r. The first terms are $r(r-1)a_0x^r$, $\alpha r a_0 x^{r+1-s}$ and $\beta a_0 x^{r+2-t}$, respectively. Thus if $s \leq 1$ and $t \leq 2$, the quadratic term will appear in the indicial equation.

5.7

1. It is clear that $x = 0$ is a singular point. The differential equation is in the standard form given in Theorem 5.6.1 with $xp(x) = 2$ and $x^2q(x) = x$. Both are analytic at $x = 0$, so $x = 0$ is a regular singular point. Substituting $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ in the differential equation, shifting indices appropriately, and collecting coefficients of like powers of x yields

$$[r(r-1)+2r]a_0x^r + \sum_{n=1}^{\infty}[(r+n)(r+n+1)a_n + a_{n-1}]x^{r+n} = 0.$$

The indicial equation is $F(r) = r(r+1) = 0$ with roots $r_1 = 0$, $r_2 = -1$. Treating a_n as a function of r, we see that $a_n(r) = -a_{n-1}(r)/F(r+n)$, $n = 1, 2, \ldots$ if $F(r+n) \neq 0$. Thus $a_1(r) = -a_0/F(r+1)$, $a_2(r) = a_0/F(r+1)F(r+2), \ldots$, and $a_n(r) = (-1)^n a_0/F(r+1)F(r+2)\ldots F(r+n)$, provided $F(r+n) \neq 0$ for $n = 1, 2, \ldots$. For the case $r_1 = 0$, we have $a_n(0) = (-1)^n a_0/F(1)F(2)\ldots F(n) = (-1)^n a_0/n!(n+1)!$, so one solution is

$$y_1(x) = \sum_{n=0}^{\infty}(-1)^n x^n/n!(n+1)!$$

where we have set $a_0 = 1$. If we try to use the above recurrence relation for the case $r_2 = -1$, we find that $a_n(-1) = -a_{n-1}/n(n-1)$, which is undefined for $n = 1$. Thus we must follow the procedure described at the end of Section 5.6 to calculate a second solution of the form given in Equation (24). Specifically, we use Equations (19) and (20) of Section 5.6 to calculate a and $c_n(r_2)$, where $r_2 = -1$. Since $r_1 - r_2 = 1 = N$, we have $a_N(r) = a_1(r) = -1/F(r+1)$, with $a_0 = 1$. Hence

$$a = \lim_{r\to-1}[(r+1)(-1)/F(r+1)] = \lim_{r\to-1}[-(r+1)/(r+1)(r+2)] = -1.$$

Next,

$$c_n(-1) = \frac{d}{dr}[(r+1)a_n(r)]|_{r=-1} = (-1)^n \frac{d}{dr}\left[\frac{r+1}{F(r+1)\dots F(r+n)}\right]|_{r=-1}$$

where we again have set $a_0 = 1$. Observe that

$(r+1)/F(r+1)\dots F(r+n) = 1/[(r+2)^2(r+3)^2\dots(r+n)^2(r+n+1)] = 1/G_n(r)$.

Hence $c_n(-1) = (-1)^{n+1}G_n'(-1)/G_n^2(-1)$. Then $G_n(-1) = 1^2 \cdot 2^2 \cdot 3^2 \dots (n-1)^2 n = (n-1)!n!$ and $G_n'(-1)/G_n(-1) = 2[1/1 + 1/2 + 1/3 + \dots + 1/(n-1)] + 1/n = H_n + H_{n-1}$. Thus $c_n(-1) = (-1)^{n+1}(H_n + H_{n-1})/(n-1)!n!$. From Equation (24) of Section 5.6 we obtain the second solution

$$y_2(x) = -y_1(x)\ln x + x^{-1}\left[1 - \sum_{n=1}^{\infty}(-1)^n(H_n + H_{n-1})x^n/n!(n-1)!\right].$$

2. It is clear that $x = 0$ is a singular point. The D.E. is in the standard form given in Theorem 5.6.1 with $xp(x) = 3$ and $x^2q(x) = 1 + x$. Both are analytic at $x = 0$, so $x = 0$ is a regular singular point. Substituting

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

in the D.E., shifting indices appropriately, and collecting coefficients of like powers of x yields

$$[r(r-1) + 3r + 1]a_0x^r + \sum_{n=1}^{\infty}\{[(r+n)(r+n+2) + 1]a_n + a_{n-1}\}x^{n+r} = 0.$$

The indicial equation is $F(r) = r^2 + 2r + 1 = (r+1)^2 = 0$ with the double root $r_1 = r_2 = -1$. Treating a_n as a function of r, we can see immediately that $a_n(r) = -a_{n-1}(r)/F(r+n)$, $n = 1, 2, \dots$. Thus we obtain that $a_1(r) = -a_0/F(r+1)$, then $a_2(r) = a_0/F(r+1)F(r+2), \dots, a_n(r) = (-1)^n a_0/F(r+1)F(r+2)\dots F(r+n)$. Setting $r = -1$, we find that $a_n(-1) = (-1)^n a_0/(n!)^2$, $n = 1, 2, \dots$. Hence one solution is

$$y_1(x) = x^{-1}\sum_{n=0}^{\infty}(-1)^n x^n/(n!)^2$$

where we have set $a_0 = 1$. To find a second solution we follow the procedure described in Section 5.6 for the case when the roots of the indicial equation are equal. Specifically, the second solution will have the form given in Eq. (17) of that section. We must calculate $a_n'(-1)$. Let us denote $G_n(r) = F(r+1)\dots F(r+n) = (r+2)^2(r+3)^2\dots(r+n+1)^2$ and take $a_0 = 1$, then $a_n'(-1) = (-1)^n/[1/G_n(r)]'$ evaluated at $r = -1$. Hence $a_n'(-1) = (-1)^{n+1}G_n'(-1)/G_n^2(-1)$. But $G_n(-1) = (n!)^2$ and $G_n'(-1)/G_n(-1) = 2[1/1 + 1/2 + 1/3 + \dots + 1/n] = 2H_n$. Thus a second solution is

$$y_2(x) = y_1(x)\ln x - 2x^{-1}\sum_{n=1}^{\infty}(-1)^n H_n x^n/(n!)^2.$$

5. Since $x = 0$ is a regular singular point, substitute

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

in the differential equation, shift indices appropriately, and collect coefficients of like powers of x to obtain

$$[r^2 - 9/4]a_0x^r + [(r+1)^2 - 9/4]a_1x^{r+1} +$$
$$+ \sum_{n=2}^{\infty} \left\{[(r+n)^2 - 9/4]a_n + a_{n-2}\right\} x^{n+r} = 0.$$

The indicial equation is $F(r) = r^2 - 9/4 = 0$ with roots $r_1 = 3/2$, $r_2 = -3/2$. Treating a_n as a function of r, we see that $a_n(r) = -a_{n-2}(r)/F(r+n)$, $n = 2, 3, \ldots$ if $F(r+n) \neq 0$. For the case $r_1 = 3/2$, $F(r_1+1)$, which is the coefficient of x^{r_1+1}, is $\neq 0$ so we must set $a_1 = 0$. It follows that $a_3 = a_5 = \ldots = 0$. For the even coefficients, set $n = 2m$ so $a_{2m}(3/2) = a_{2m-2}(3/2)/F(3/2+2m) = -a_{2m-2}/2^2m(m+3/2)$, $m = 1, 2, \ldots$. Thus $a_2(3/2) = -a_0/2^2 \cdot 1(1+3/2)$, $a_4(3/2) = a_0/2^4 \cdot 2!(1+3/2)(2+3/2), \ldots$, and $a_{2m}(3/2) = (-1)^m/2^{2m}m!(1+3/2)\ldots(m+3/2)$. Hence one solution is

$$y_1(x) = x^{3/2}\left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{m!(1+3/2)(2+3/2)\ldots(m+3/2)}\left(\frac{x}{2}\right)^{2m}\right],$$

where we have set $a_0 = 1$. For this problem, the roots r_1 and r_2 of the indicial equation differ by an integer: $r_1 - r_2 = 3$. Hence we can anticipate that there may be difficulty in calculating a second solution corresponding to $r = r_2$. This difficulty will occur in calculating $a_3(r) = -a_1(r)/F(r+3)$ since when $r = r_2 = -3/2$ we have $F(r_2+3) = F(r_1) = 0$. However, in this problem we are fortunate because $a_1 = 0$ and it will not be necessary to use the theory described at the end of Section 5.6. Notice for $r = r_2 - 3/2$ that the coefficient of x^{r_2+1} is $[(r_2+1)^2 - 9/4]a_1$, which does not vanish unless $a_1 = 0$. Thus the recurrence relation for the odd coefficients yields $a_5 = -a_3/F(7/2)$, $a_7 = -a_5/F(11/2) = a_3/F(11/2)F(7/2)$, and so forth. Substituting these terms into the assumed form, we see that a multiple of $y_1(x)$ has been obtained and thus we may take $a_3 = 0$ without loss of generality. Hence $a_3 = a_5 = a_7 = \ldots = 0$. The even coefficients are given by $a_{2m}(-3/2) = -a_{2m-2}(-3/2)/F(2m-3/2)$, $m = 1, 2, \ldots$. Thus $a_2(-3/2) = -a_0/2^2 \cdot 1 \cdot (1-3/2)$, $a_4(-3/2) = a_0/2^4 \cdot 2!(1-3/2)(2-3/2), \ldots$, and $a_{2m}(-3/2) = (-1)^m a_0/2^{2m}m!(1-3/2)(2-3/2)\ldots(m-3/2)$. Thus a second solution is

$$y_2(x) = x^{-3/2}\left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{m!(1-3/2)(2-3/2)\ldots(m-3/2)}\left(\frac{x}{2}\right)^{2m}\right].$$

7. Apply the ratio test:

$$\lim_{m\to\infty} \frac{|(-1)^{m+1}x^{2m+2}/2^{2m+2}[(m+1)!]^2|}{|(-1)^m x^{2m}/2^{2m}(m!)^2|} = |x^2| \lim_{m\to\infty} \frac{1}{2^2(m+1)^2} = 0$$

for every x. Thus the series for $J_0(x)$ converges absolutely for all x.

12. Consider a solution of the form $y(x) = \sqrt{x}\; f(\alpha x^\beta)$. Then

$$y' = \frac{df}{d\xi} \cdot \frac{\alpha\beta\, x^\beta}{\sqrt{x}} + \frac{f(\xi)}{2\sqrt{x}}$$

in which $\xi = \alpha x^\beta$. Hence

$$y'' = \frac{d^2 f}{d\xi^2} \cdot \frac{\alpha^2\beta^2\, x^{2\beta}}{x\sqrt{x}} + \frac{df}{d\xi} \cdot \frac{\alpha\beta^2\, x^\beta}{x\sqrt{x}} - \frac{f(\xi)}{4x\sqrt{x}}\,,$$

and

$$x^2\, y'' = \alpha^2\beta^2\, x^{2\beta}\sqrt{x}\; \frac{d^2 f}{d\xi^2} + \alpha\beta^2\, x^\beta\sqrt{x}\; \frac{df}{d\xi} - \frac{1}{4}\sqrt{x}\; f(\xi)\,.$$

Substitution into the ODE results in

$$\alpha^2\beta^2\, x^{2\beta}\, \frac{d^2 f}{d\xi^2} + \alpha\beta^2\, x^\beta\, \frac{df}{d\xi} - \frac{1}{4}\, f(\xi) + (\alpha^2\beta^2\, x^{2\beta} + \frac{1}{4} - \nu^2\beta^2) f(\xi) = 0\,.$$

Simplifying, and setting $\xi = \alpha x^\beta$, we find that

$$\xi^2\, \frac{d^2 f}{d\xi^2} + \xi\, \frac{df}{d\xi} + (\xi^2 - \nu^2) f(\xi) = 0\,, \quad (*)$$

which is a Bessel equation of order ν . Therefore, the general solution of the given ODE is

$$y(x) = \sqrt{x}\; \left[c_1\, f_1(\alpha x^\beta) + c_2\, f_2(\alpha x^\beta)\right],$$

in which $f_1(\xi)$ and $f_2(\xi)$ are the linearly independent solutions of $(*)$.

13. To compare $y'' - xy = 0$ with the differential equation of Problem 12, we must multiply by x^2 to get $x^2y'' - x^3y = 0$. Thus $2\beta = 3$, $\alpha^2\beta^2 = -1$ and $1/4 - \nu^2\beta^2 = 0$. Hence $\beta = 3/2$, $\alpha = 2i/3$, and $\nu^2 = 1/9$, which yields the desired result.

14. First we verify that $J_0(\lambda_j x)$ satisfies the D.E. We know that $J_0(t)$ is a solution of the Bessel equation of order zero:

$$t^2 J_0''(t) + tJ_0'(t) + t^2 J_0(t) = 0 \;\text{ or}$$
$$J_0''(t) + t^{-1}J_0'(t) + J_0(t) = 0.$$

Let $t = \lambda_j x$. Then

$$\frac{d}{dx}J_0(\lambda_j x) = \frac{d}{dt}J_0(t)\frac{dt}{dx} = \lambda_j J_0'(t),$$
$$\frac{d^2}{dx^2}J_0(\lambda_j x) = \lambda_j \frac{d}{dt}[J_0'(t)]\frac{dt}{dx} = \lambda_j^2 J_0''(t).$$

Substituting $y = J_0(\lambda_j x)$ in the given D.E. and making use of these results, we have

$$\lambda_j^2 J_0''(t) + (\lambda_j/t)\lambda_j J_0'(t) + \lambda_j^2 J_0(t) = \lambda_j^2\left[J_0''(t) + t^{-1}J_0'(t) + J_0(t)\right] = 0.$$

Thus $y = J_0(\lambda_j x)$ is a solution of the given D.E. For the second part of the problem we follow the hint. First, rewrite the D.E. by multiplying by x to yield $xy'' + y' + \lambda_j^2 xy = 0$, which can be written as $(xy')' = -\lambda_j^2 xy$. Now let $y_i(x) = J_0(\lambda_i x)$ and $y_j(x) = J_0(\lambda_j x)$ and we have, respectively, $(xy_i')' = -\lambda_i^2 xy_i$ and $(xy_j')' = -\lambda_j^2 xy_j$.

Now, multiply the first equation by y_j , the second by y_i, integrate each from 0 to 1, and subtract the second from the first:

$$\int_0^1 \left[y_j \left(xy_i'\right)' - y_i \left(xy_j'\right)'\right] dx = -\left(\lambda_i^2 - \lambda_j^2\right) \int_0^1 xy_iy_j dx.$$

If we integrate each term on the left side once by parts and note that $y_i = y_j = 0$ and $x = 1$, we find that the left side of this equation is identically zero. Hence the right side is identically zero and for $\lambda_i \neq \lambda_j$ this gives the desired result.