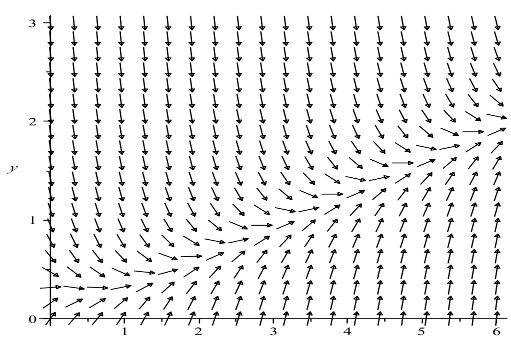
CHAPTER
2

First Order Differential Equations

2.1

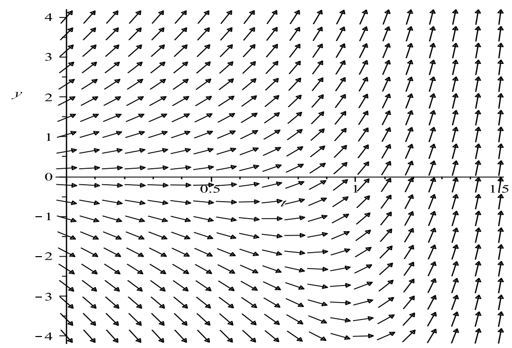
1.(a)



(b) All solutions seem to approach a line in the region where the negative and positive slopes meet each other.

(c) $\mu(t) = e^{\int 3\,dt} = e^{3t}$. Thus $e^{3t}(y' + 3y) = e^{3t}(t + e^{-2t})$ or $(ye^{3t})' = te^{3t} + e^t$. Integration of both sides yields $ye^{3t} = te^{3t}/3 - e^{3t}/9 + e^t + c$, where integration by parts is used on the right side, with $u = t$ and $dv = e^{3t}dt$. Division by e^{3t} gives $y(t) = ce^{-3t} + t/3 - 1/9$, so y approaches $t/3 - 1/9$ as $t \to \infty$. This is the line identified in part (b).

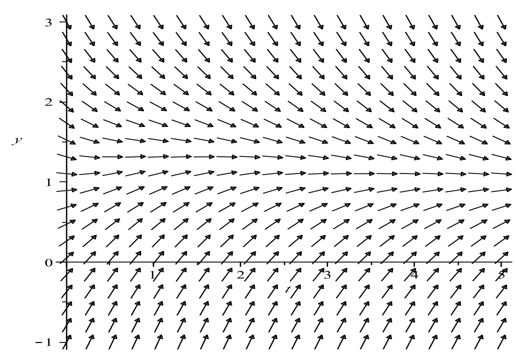
2.(a)



(b) All solutions eventually have positive slopes, and hence increase without bound.

(c) The integrating factor is $\mu(t) = e^{-2t}$, and hence $y(t) = t^3 e^{2t}/3 + c\, e^{2t}$. It is evident that all solutions increase at an exponential rate.

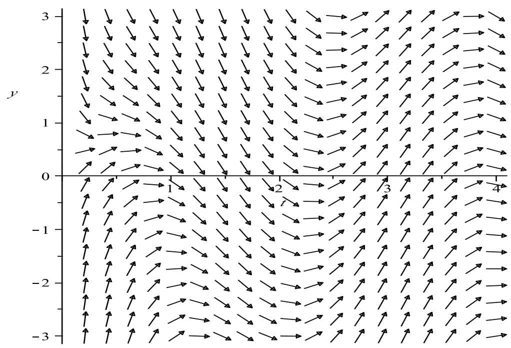
3.(a)



(b) All solutions seem to converge to the function $y_0(t) = 1$.

(c) The integrating factor is $\mu(t) = e^t$, and hence $y(t) = t^2 e^{-t}/2 + 1 + c\, e^{-t}$. It is clear that all solutions converge to the specific solution $y_0(t) = 1$.

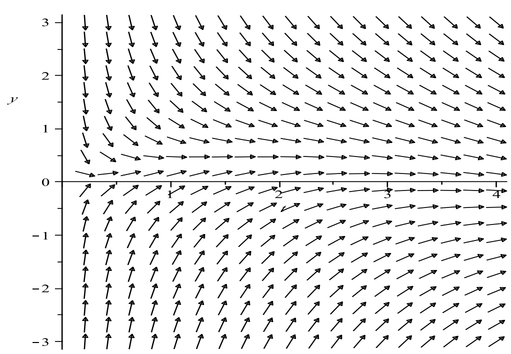
4.(a)



(b) Based on the direction field, the solutions eventually become oscillatory.

(c) The integrating factor is $\mu(t) = e^{\int(1/t)\,dt} = e^{\ln t} = t$, so $(ty)' = 3t\cos 2t$, and integration by parts yields the general solution $y(t) = (3/4t)\cos 2t + (3/2)\sin 2t + c/t$, in which c is an arbitrary constant. As t becomes large, all solutions converge to the function $y_1(t) = 3(\sin 2t)/2$.

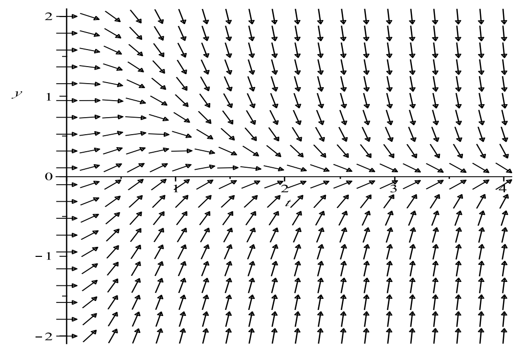
6.(a)



(b) All solutions seem to converge to the function $y_0(t) = 0$.

(c) The equation must be divided by t so that it is in the form of Eq.(3): $y' + (2/t)y = \sin t/t$. Thus $\mu(t) = e^{\int(2/t)\,dt} = t^2$, and $(t^2y)' = t\sin t$. Integration then yields $t^2y = -t\cos t + \sin t + c$, hence the general solution is $y(t) = -(\cos t)/t + (\sin t)/t^2 + c/t^2$, in which c is an arbitrary constant $(t > 0)$. As t becomes large, all solutions converge to the function $y_0(t) = 0$.

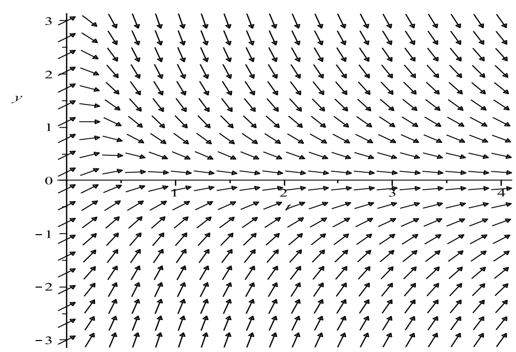
7.(a)



(b) All solutions seem to converge to the function $y_0(t) = 0$.

(c) The integrating factor is $\mu(t) = e^{t^2}$, and hence $y(t) = t^2 e^{-t^2} + c\, e^{-t^2}$. It is clear that all solutions converge to the function $y_0(t) = 0$.

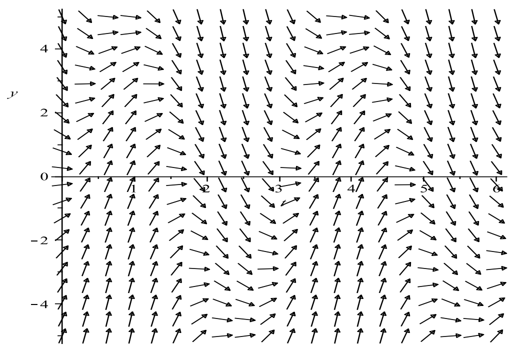
8.(a)



(b) All solutions seem to converge to the function $y_0(t) = 0$.

(c) Since $\mu(t) = e^{\int 4t/(1+t^2)\, dt} = (1+t^2)^2$, after integration we obtain the general solution $y(t) = (\arctan t + c)/(1+t^2)^2$. It follows that all solutions converge to the function $y_0(t) = 0$.

11.(a)



(b) The solutions appear to be oscillatory.

(c) The integrating factor is $\mu(t) = e^t$, so $(e^t y)' = 5e^t \sin 2t$. To integrate the right side we can integrate by parts (twice), use an integral table or use a symbolic computational software to find $y(t) = \sin 2t - 2\cos 2t + c\,e^{-t}$. It is evident that all solutions converge to the specific solution $y_0(t) = \sin 2t - 2\cos 2t$.

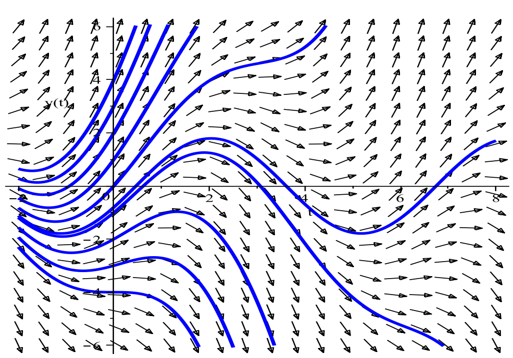
13. $\mu(t) = e^{-t}$, so $(e^{-t}y)' = 2te^t$ and thus $e^{-t}y = 2\int te^t\,dt + c = 2(te^t - \int e^t\,dt) + c = 2(te^t - e^t) + c$. Thus $y(t) = 2(t-1)e^{2t} + ce^t$, so setting $t = 0$ we have $1 = -2 + c$, or $c = 3$. Hence $y(t) = 2(t-1)e^{2t} + 3e^t$.

15. $\mu(t) = e^{\int (2/t)\,dt} = t^2$ so that $(t^2 y)' = t^3 - t^2 + t$. Integrating and dividing by t^2 gives $y = t^2/4 - t/3 + 1/2 + c/t^2$. Setting $t = 1$ and $y = 1/2$ we have $c = 1/12$.

18. $\mu(t) = t^2$. Thus $(t^2 y)' = t \sin t$ and $t^2 y = -t\cos t + \sin t + c$. Setting $t = \pi/2$ and $y = 1$ yields $c = \pi^2/4 - 1$.

20. $\mu(t) = e^{\int (t+1)/t\,dt} = e^{1+\ln t} = te^t$.

21.(a)

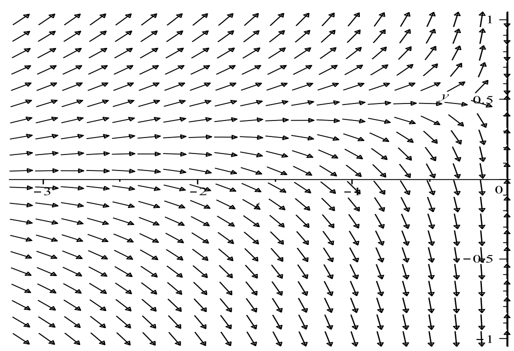


The solutions appear to diverge from an apparent oscillatory solution. From the direction field, the critical value of the initial condition seems to be $a_0 = -1$. For $a > -1$, the solutions increase without bound. For $a < -1$, solutions decrease without bound.

(b) The integrating factor is $\mu(t) = e^{-t/2}$, so $(e^{-t/2}y)' = 2e^{-t/2}\cos t$. Integrating (see comments in Problem 11) and dividing by $e^{-t/2}$ yields the general solution $y(t) = (8\sin t - 4\cos t)/5 + c\,e^{t/2}$. Thus $y(0) = -4/5 + c = a$, or $c = a + 4/5$ and $y(t) = -4\cos t/5 + 8\sin t/5 + (a + 4/5)e^{t/2}$.

(c) If $a + 4/5 = 0$, then the solution is oscillatory for all t, while if $a + 4/5 \neq 0$, the solution is unbounded as $t \to \infty$. Thus $a_0 = -4/5$.

25.(a)

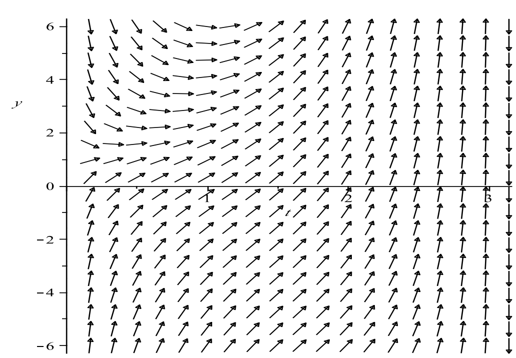


As $t \to 0$, solutions increase without bound if $y(-\pi/2) = a > 0.4$, and solutions decrease without bound if $y(-\pi/2) = a < 0.4$.

(b) $\mu(t) = e^{\int (2/t)\,dt} = t^2$, so $(t^2y)' = \sin t$ and $y(t) = -\cos t/t^2 + c/t^2$. Setting $t = -\pi/2$ yields $4c/\pi^2 = a$ or $c = a\pi^2/4$ and hence $y(t) = (a\,\pi^2/4 - \cos\,t)/t^2$, which is unbounded as $t \to 0$ unless $a\pi^2/4 = 1$. Since $\lim_{t\to 0}\cos\,t = 1$, solutions increase without bound if $a > 4/\pi^2$, and solutions decrease without bound if $a < 4/\pi^2$. Hence the critical value is $a_0 = 4/\pi^2 \approx 0.452847$.

(c) For $a_0 = 4/\pi^2$, the solution is $y(t) = (1 - \cos\,t)/t^2$, and using L'Hospital's rule twice we obtain $\lim_{t\to 0} y(t) = 1/2$. Hence the solution is bounded.

26.(a)



(b) $\mu(t) = e^{\int \cos t/\sin t\,dt} = e^{\ln \sin t} = \sin t$ and thus $(y \sin t)' = e^t$. Hence $y \sin t = e^t + c$ or $y = (e^t + c)/\sin t$. Setting $t = 1$ and $y = a$ we get $c = a \sin 1 - e$ so $y(t) = (e^t - e + a \sin 1)/\sin t$. If $y(t)$ is to remain finite as $t \to 0$ the numerator, $e^t - e + a \sin 1$, must approach 0 as $t \to 0$ and hence $a_0 = (e-1)/\sin 1$.

(c) Using a_0 we have $y(t) = (e^t - 1)/\sin t$, which approaches 1 as $t \to 0$, using L'Hospital's rule.

30. $(e^{-t}y)' = e^{-t} + 3e^{-t}\sin t$ so $e^{-t}y = -e^{-t} - 3e^{-t}(\sin t + \cos t)/2 + c$ or $y(t) = -1 - 3(\sin t + \cos t)/2 + ce^t$. Thus $y(0) = -1 - 3/2 + c = y_0$ or $c = y_0 + 5/2$. Now, if $y(t)$ is to remain bounded as $t \to \infty$, we must have $c = 0$ so that $y_0 = -5/2$.

32. Write the first term of Eq.(47) as $(\int_0^t e^{s^2/4}\,ds)/e^{t^2/4}$. In applying L'Hospital's rule, the derivative of the numerator term is $e^{t^2/4}$ by the Fundamental Theorem of Calculus. The derivative of the denominator is $(t/2)e^{t^2/4}$ and thus the limit of both terms in Eq.(47) is 0 as $t \to \infty$.

33. $\mu(t) = e^{at}$ so the differential equation can be written as $(e^{at}y)' = be^{at}e^{-\lambda t} = be^{(a-\lambda)t}$. If $a \neq \lambda$, then integration and solution for y yields $y = [b/(a-\lambda)]e^{-\lambda t} + ce^{-at}$. Then $\lim_{t\to\infty} y$ is zero since both λ and a are positive numbers. If $a = \lambda$, then the differential equation becomes $(e^{at}y)' = b$, which yields $y = (bt + c)e^{-\lambda t}$ as the solution. L'Hospital's rule gives

$$\lim_{t\to\infty} y = \lim_{t\to\infty} \frac{bt + c}{e^{\lambda t}} = \lim_{t\to\infty} \frac{b}{\lambda e^{\lambda t}} = 0.$$

35. There is no unique answer for this situation. One possible answer is to assume $y(t) = ce^{-2t} + 3 - t$ (which satisfies the given condition), then $y'(t) = -2ce^{-2t} - 1$. Eliminating ce^{-2t} between the two equations yields $y' + 2y = 5 - 2t$.

39. By Eq.(iii), Problem 38, $y(t) = A(t)e^{-\int(-2)\,dt} = A(t)e^{2t}$. Differentiating $y(t)$ and substituting into the differential equation yields $A'(t) = t^2$ since the terms involving $A(t)$ add to zero. Thus $A(t) = t^3/3 + c$, which substituted into $y(t)$ yields the solution.

42. Since $p(t) = 1/2$, $y(t) = A(t)e^{-\int(1/2)\,dt} = A(t)e^{-t/2}$ and $A'(t) = (3/2)t^2e^{t/2}$. Integration of $A'(t)$ and substituting in $y(t)$ yields the desired solution.

2.2

Problems 1 through 20 follow the pattern of the examples worked in this section. The first eight problems, however, do not have an initial condition, so the integration constant c cannot be found.

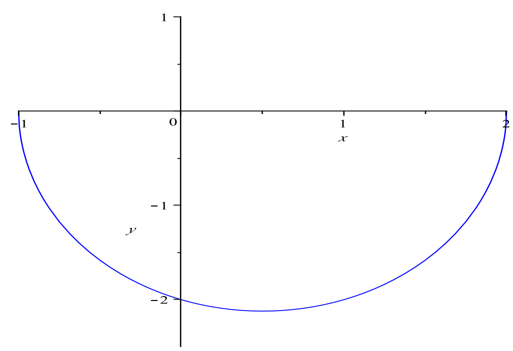
1. Write the equation in the form $ydy = x^2dx$. Integrating the left side with respect to y and the right side with respect to x yields $y^2/2 = x^3/3 + C$, or $3y^2 - 2x^3 = c$.

4. For $y \neq -3/2$ multiply both sides of the equation by $3 + 2y$ to get the separated equation $(3 + 2y)dy = (3x^2 - 1)dx$. Integration then yields $3y + y^2 = x^3 - x + c$.

6. We need $x \neq 0$ and $|y| < 1$ for this problem to be defined. Separating the variables we get $(1 - y^2)^{-1/2}\, dy = x^{-1}\, dx$. Integrating each side yields $\arcsin y = \ln|x| + c$, so $y = \sin(\ln|x| + c)$, $x \neq 0$ (note that $|y| < 1$). Also, $y = \pm 1$ satisfy the differential equation, since both sides are zero.

10.(a) Separating the variables we get $ydy = (1 - 2x)dx$, so $y^2/2 = x - x^2 + c$. Setting $x = 1$ and $y = -2$ we have $c = 2$ and thus $y^2 = 2x - 2x^2 + 4$ or $y(x) = -\sqrt{2x - 2x^2 + 4}$. The negative square root must be used since $y(1) = -2$.

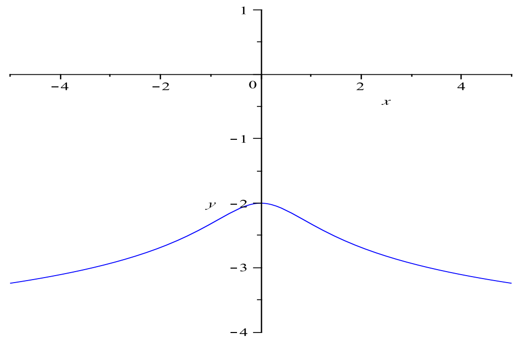
(b)



(c) Rewriting $y(x)$ as $y(x) = -\sqrt{2(2 - x)(x + 1)}$, we see that y is defined for $-1 \leq x \leq 2$. However, since y' does not exist for $x = -1$ or $x = 2$, the solution is valid only for the open interval $-1 < x < 2$. The interval of existence is $(-1, 2)$.

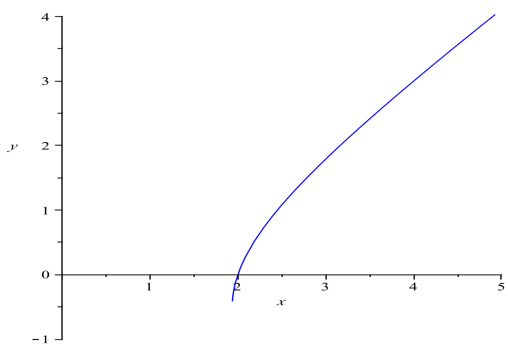
13.(a) Separate variables by factoring the denominator of the right side to get $ydy = 2x/(1 + x^2)\, dx$. Integration yields $y^2/2 = \ln(1 + x^2) + c$ and use of the initial condition gives $c = 2$. Thus $y = \pm\sqrt{2\ln(1 + x^2) + 4}$, but we must discard the plus square root because of the initial condition. Since $1 + x^2 > 0$, the solution is valid for all x.

(b)



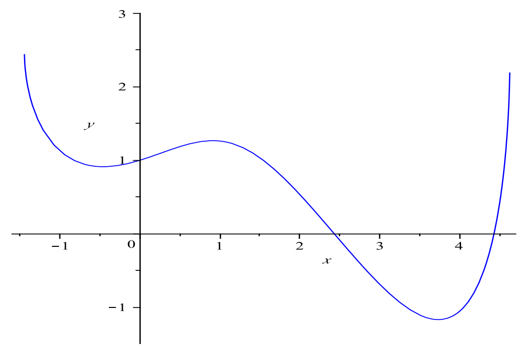
15.(a) Separating variables and integrating yields $y + y^2 = x^2 + c$. Setting $y = 0$ when $x = 2$ yields $c = -4$ or $y^2 + y = x^2 - 4$. To solve for y complete the square on the left side by adding $1/4$ to both sides. This yields $y^2 + y + 1/4 = x^2 - 4 + 1/4$ or $(y + 1/2)^2 = x^2 - 15/4$. Taking the square root of both sides yields $y + 1/2 = \pm\sqrt{x^2 - 15/4}$, where the positive square root must be taken in order to satisfy the initial condition. Thus $y(x) = -1/2 + \sqrt{x^2 - 15/4}$, which is defined for $x^2 \geq 15/4$ or $x \geq \sqrt{15}/2$.

(b)



17.(a) Separating variables gives $(2y - 5)dy = (3x^2 - e^x)dx$ and integration then gives $y^2 - 5y = x^3 - e^x + c$. Setting $x = 0$ and $y = 1$ we have $1 - 5 = 0 - 1 + c$, or $c = -3$ and thus $y^2 - 5y - (x^3 - e^x - 3) = 0$. Using the quadratic formula then gives $y(x) = 5/2 - \sqrt{x^3 - e^x + 13/4}$, where the negative square root is chosen so that $y(0) = 1$.

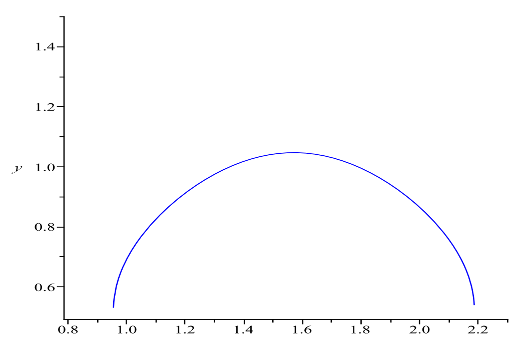
(b)



(c) The solution is valid for approximately $-1.45 < x < 4.63$. These values are found by estimating the roots of $4x^3 - 4e^x + 13 = 0$.

19.(a) We start with $\cos 3y\, dy = -\sin 2x\, dx$ and integrate to get $(1/3)\sin 3y = (1/2)\cos 2x + c$. Setting $y = \pi/3$ when $x = \pi/2$ (from the initial condition) we find that $0 = -1/2 + c$ or $c = 1/2$, so that $(1/3)\sin 3y = (1/2)\cos 2x + 1/2 = \cos^2 x$ (using the appropriate trigonometric identity). To solve for y we must choose the branch that passes through the point $(\pi/2, \pi/3)$, so $y(x) = (\pi - \arcsin(3\cos^2 x))/3$.

(b)



(c) The solution in part (a) is defined only for $0 \le 3\cos^2 x \le 1$, or $-\sqrt{1/3} \le \cos x \le \sqrt{1/3}$. Taking the indicated square roots and then finding the inverse cosine of each side yields $0.9553 \le x \le 2.1863$, or $|x - \pi/2| \le 0.6155$, as the appropriate interval.

21. We have $(3y^2 - 6y)dy = (1 + 3x^2)dx$ so that $y^3 - 3y^2 = x + x^3 - 2$, once the initial condition is used. From the differential equation, the integral curve will have a vertical tangent when $3y^2 - 6y = 0$, or $y = 0, 2$. For $y = 0$ we have $x^3 + x - 2 = 0$, which is satisfied for $x = 1$, which is the only zero of the function $w = x^3 + x - 2$. Likewise, for $y = 2$, $x = -1$. Thus the solution is valid on $|x| < 1$.

23. Separating variables gives $y^{-2}\, dy = (2 + x)dx$, and after integration we get $-y^{-1} = 2x + x^2/2 + c$. $y(0) = 1$ yields $c = -1$, and thus $y = 2/(2 - 4x - x^2)$. This

implies that $y'(x) = (8+4x)/(2-4x-x^2)^2$, so the minimum value is attained at $x = -2$. Note that the solution is defined for $-2-\sqrt{6} < x < -2+\sqrt{6}$ (by finding the zeros of the denominator) and has vertical asymptotes at the endpoints of the interval.

25. Separating variables and integrating yields $3y + y^2 = \sin 2x + c$. $y(0) = -1$ gives $c = -2$ so that $y^2 + 3y + 2 - \sin 2x = 0$. The quadratic formula, along with the initial condition then gives $y = -3/2 + \sqrt{\sin 2x + 1/4}$, which is defined for $-0.126 < x < 1.697$ (found by solving $\sin 2x = -1/4$ for x and noting $x = 0$ is the initial point). Thus we have $y' = \cos 2x/\sqrt{\sin 2x + 1/4}$, which yields $x = \pi/4$ as the only critical point in the above interval. Using the second derivative test or graphing the solution indicates the critical point is a maximum.

27.(a) By sketching the direction field or by using the differential equation we note that $y' < 0$ for $y > 4$ and y' approaches zero as y approaches 4. For $0 < y < 4$, $y' > 0$ and again approaches 0 as y approaches 4. Thus $\lim_{t\to\infty} y = 4$ if $y_0 > 0$. For $y_0 < 0$, $y' < 0$ for all y and hence y becomes negatively unbounded as t increases. If $y_0 = 0$, then $y' = 0$ for all t, so $y = 0$ for all t.

(b) Separating variables and using a partial fraction expansion we obtain that $(1/y - 1/(y-4))\,dy = (4/3)t\,dt$. Hence $\ln|y/(y-4)| = 2t^2/3 + c_1$, and therefore $|y/(y-4)| = e^{c_1}e^{2t^2/3} = ce^{2t^2/3}$, where c is positive. For $y(0) = y_0 = 0.5$, this gives us the equation $|0.5/(0.5-4)| = c$ and thus $c = 1/7$. Using this value for c and solving for y yields $y(t) = 4/(1 + 7e^{-2t^2/3})$. Setting this equal to 3.98 and solving for t yields $t = 3.29527$.

29. Separating variables yields $(cy+d)/(ay+b)\,dy = dx$. If $a \neq 0$ and $ay + b \neq 0$ then $dx = (c/a + (ad-bc)/(a(ay+b))\,dy$. Integration then yields the desired answer.

30.(a) Divide the numerator and denominator by $x \neq 0$.

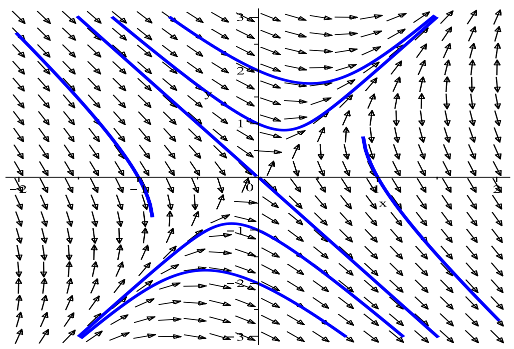
(b) If $v = y/x$, then $y = vx$ and $dy/dx = v + xdv/dx$.

(c) The differential equation becomes $v + xdv/dx = (v-4)/(1-v)$. Subtracting v from both sides yields $xdv/dx = (v^2-4)/(1-v)$.

(d) The last equation in part (c) separates into $(1-v)/(v^2-4)dv = (1/x)dx$. To integrate the left side use partial fractions to write $(1-v)/(v^2-4) = A/(v-2) + B/(v+2)$, which yields $A = -1/4$ and $B = -3/4$. Integrating both sides gives $-(1/4)\ln|v-2| - (3/4)\ln|v+2| = \ln|x| - k$, or $\ln|x^4||v-2||v+2|^3 = 4k$ after manipulations using properties of the logarithmic function. Thus $x^4|v-2||v+2|^3 = c$.

(e) Recalling that $v = y/x$ gives the desired solution.

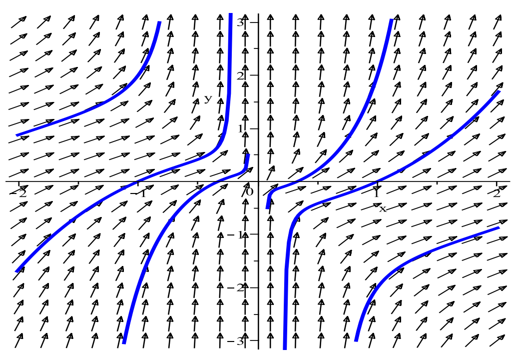
(f)



31.(a) Simplifying the right side of the differential equation gives $dy/dx = 1 + (y/x) + (y/x)^2$ so the equation is homogeneous.

(b) $y = vx$ gives $y' = v + xv'$, so substitution leads to the equation $v + xv' = 1 + v + v^2$ or $(1/(1+v^2))\,dv = (1/x)\,dx$. Integrating, we get $\arctan v = \ln|x| + c$ and substituting for v we obtain $\arctan(y/x) - \ln|x| = c$.

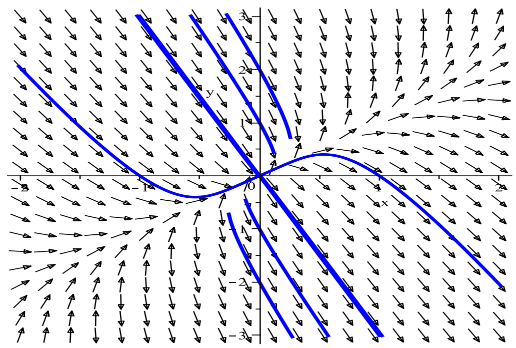
(c) The integral curves are symmetric with respect to the origin.



33.(a) Observe that $(4y - 3x)/(2x - y) = (4(y/x) - 3)/(2 - y/x)$. Hence the differential equation is homogeneous.

(b) Substituting $y = vx$ we get $v + xv' = (4v - 3)/(2 - v)$ which can be rewritten as $xv' = (v^2 + 2v - 3)/(2 - v)$. Note that $v = -3$ and $v = 1$ are solutions of this equation. For $v \neq 1, -3$ separating variables gives $(2 - v)/((v + 3)(v - 1))\,dv = (1/x)\,dx$. Applying a partial fraction decomposition to the left side we obtain $(1/(4(v - 1)) - 5/(4(v + 3)))\,dv = (1/x)\,dx$, and upon integrating both sides we find that $(1/4)\ln|v - 1| - (5/4)\ln|v + 3| = \ln|x| + c$. Substituting for v and performing some algebraic manipulations we get the solution in the implicit form $|y - x| = c|y + 3x|^5$. $v = 1$ and $v = -3$ yield $y = x$ and $y = -3x$, respectively, as solutions also.

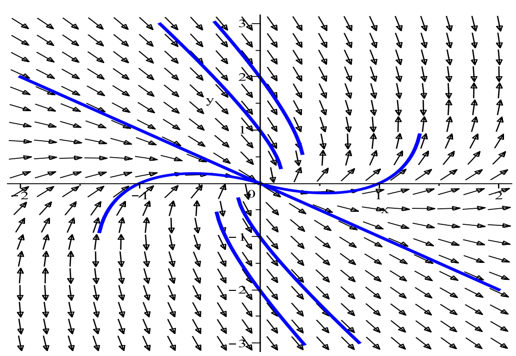
(c) The integral curves are symmetric with respect to the origin.



35.(a) Observe that $(x+3y)/(x-y) = (1+3(y/x))/(1-y/x)$. Hence the differential equation is homogeneous.

(b) Let $y = vx$; we get $v + xv' = (1+3v)/(1-v)$, so $xv' = (v+1)^2/(1-v)$. Note that $v = -1$ (or $y = -x$) satisfies the differential equation. Separating variables yields $((1-v)/(v+1)^2)\,dv = (1/x)\,dx$. Integrating the left side by parts (by letting let $u = 1-v$ and $dw = dv/(v+1)^2$) we get the equation $(v-1)/(v+1) - \ln|v+1| = \ln|x| + c$. Letting $v = y/x$ we obtain the equation $(y-x)/(y+x) - \ln|(y+x)/x| = \ln|x| + c$, or $(y-x)/(y+x) - \ln|y+x| = c$. The answer in the text can be obtained by integrating the left side above using partial fractions. By differentiating both answers it can be verified that indeed both forms satisfy the differential equation.

(c) The integral curves are symmetric with respect to the origin.



2.3

2. Let $S(t)$ be the amount of salt that is present at any time t, then $S(0) = 0$ is the original amount of salt in the tank, 2γ is the amount of salt entering per minute, and $2(S/120)$ is the amount of salt leaving per minute (all amounts measured in

grams). Thus $dS/dt = 2\gamma - 2S/120$, $S(0) = 0$. This is a linear equation, which has $e^{t/60}$ as its integrating factor. Thus the general solution is $S(t) = 120\gamma + ce^{-t/60}$. $S(0) = 0$ gives $c = -120\gamma$, so $S(t) = 120\gamma(1 - e^{-t/60})$ and hence $S(t) \to 120\gamma$ grams as $t \to \infty$.

3. We must first find the amount of salt that is present after 10 minutes. For the first 10 minutes (if we let $Q(t)$ be the amount of salt in the tank): $dQ/dt = (1/2)(2) - 2Q(t)/100$, $Q(0) = 0$. This is a linear equation which has the solution $Q(t) = 50(1 - e^{-t/50})$, as in problem 2, and thus $Q(10) = 50(1 - e^{-0.2}) = 9.063$ lbs of salt in the tank after the first 10 minutes. At this point no more salt is allowed to enter, so the new initial value problem (letting $P(t)$ be the amount of salt in the tank after the first 10 minutes) is $dP/dt = (0)(2) - 2P(t)/100$, $P(0) = Q(10) = 9.063$. The solution of this problem is $P(t) = 9.063e^{-0.02t}$, which yields $P(10) = 7.42$ lbs.

4. Salt flows out of the tank at the rate of $(Q(t)/(200 + t))(2)$ lb/min, since the volume of water in the tank at any time t is $200 + (1)(t)$ gallons (due to the fact that water flows into the tank faster than it flows out). Thus the initial value problem is $dQ/dt = (3)(1) - 2Q(t)/(200 + t)$, $Q(0) = 100$, which is a linear equation with $(200 + t)^2$ as its integrating factor.

8.(a) Set $S_0 = 0$ in Eq.(16) (or solve Eq.(15) with $S(0) = 0$).

(b) Set $r = 0.075$, $t = 40$ and $S(t) = \$1,000,000$ in the answer to part (a) and then solve for k.

(c) Set $k = \$2,000$, $t = 40$ and $S(t) = \$1,000,000$ in the answer to part (a) and then solve numerically for r.

9. Let $S(t)$ be the amount of the loan remaining at time t, then $dS/dt = 0.1S - k$, $S(0) = \$8,000$. Solving this for $S(t)$ yields $S(t) = 8000e^{0.1t} - 10k(e^{0.1t} - 1)$. Setting $S = 0$ and substitution of $t = 3$ gives $k = \$3,086.64$ per year. For 3 years this totals $\$9,259.92$, so $\$1,259.92$ has been paid in interest.

10. Since we are assuming continuity, either convert the monthly payment into an annual payment or convert the yearly interest rate into a monthly interest rate for 240 (or 360) months. Then proceed as in Problem 9.

11.(a) The monthly interest rate is 0.5%. Then $dS/dt = 0.005S - k$, $S(0) = 250,000$. The solution of this linear equation is $S(t) = 250,000e^{0.005t} + 200k(1 - e^{0.005t})$. For 20 years, we need $S(240) = 0$, and we obtain $k = \$1788.77$. For 30 years, we need that $S(360) = 0$ and we obtain $k = \$1497.54$.

(b) The total interest paid in the 20 year case is $\$179,305$, in the 30 year case is $\$289,114$.

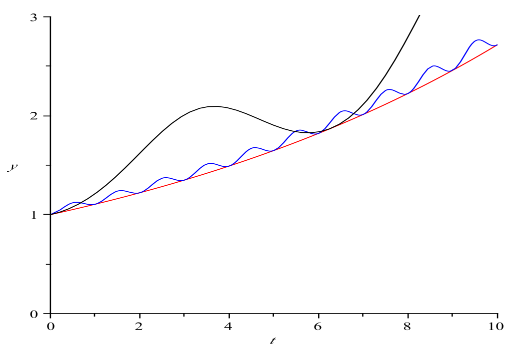
14.(a) We have $(1/y)dy = (0.1 + 0.2\sin t)dt$, by separating variables, and thus $y(t) = ce^{0.1t - 0.2\cos t}$. $y(0) = 1$ gives $c = e^{0.2}$, so $y(t) = e^{0.2 + 0.1t - 0.2\cos t}$. Setting $y = 2$ yields $\ln 2 = 0.2 + 0.1\tau - 0.2\cos\tau$, which can be solved numerically to give $\tau =$

2.9632. If $y(0) = y_0$ then as above, $y(t) = y_0 e^{0.2+0.1t-0.2\cos t}$. Thus if we set $y = 2y_0$ we get the same numerical equation for τ and hence the doubling time has not changed.

(b) The differential equation is $dy/dt = y/10$, with solution $y(t) = y(0)e^{t/10}$. The doubling time is given by $\tau = 10\ln 2 \approx 6.9315$.

(c) Consider the differential equation $dy/dt = (0.5 + \sin(2\pi t))\, y/5$. The equation is separable, with $(1/y)dy = (0.1 + \frac{1}{5}\sin(2\pi t))dt$. Integrating both sides, with respect to the appropriate variable, we obtain $\ln y = (\pi t - \cos(2\pi t))/10\pi + c$. Invoking the initial condition, the solution is $y(t) = e^{(1+\pi t-\cos(2\pi t))/10\pi}$. The doubling-time is $\tau \approx 6.3804$. The doubling time approaches the value found in part (b).

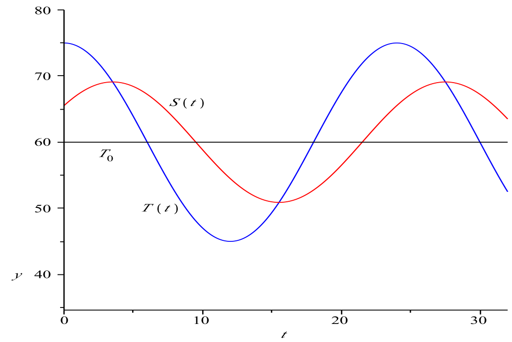
(d)



16. If T is the temperature of the coffee at any time t, then $dT/dt = -k(T - 70)$, $T(0) = 200$, $T(1) = 190$. The solution of this linear equation will involve k (the cooling rate) and the integration constant c. Use $T(0) = 200$ to find c and then use $T(1) = 190$ to evaluate k.

18.(a) Eq.(i) is a linear equation with the integrating factor e^{kt}. Thus $(e^{kt}u)' = k(T_0 + T_1\cos\omega t)e^{kt}$ and hence $e^{kt}u = T_0 e^{kt} + kT_1\int \cos\omega t e^{kt}\,dt + c$. Evaluating the integral (by parts or by a symbolic software package) and dividing by e^{kt} yields $u(t) = T_0 + kT_1(k\cos\omega t + \omega\sin\omega t)/(k^2+\omega^2) + ce^{-kt}$. Note that the last term approaches zero as $t \to \infty$ for any initial condition, and that the rest of the solution oscillates about $u(t) = T_0$.

(b) $R \approx 9°\text{F}$, $\tau \approx 3.5\text{h}$.



(c) Recall that $R\cos(\omega(t-\tau)) = R\cos\omega t\cos\omega\tau + R\sin\omega t\sin\omega\tau$. Comparing this with the oscillatory portion of the above solution we have $R\cos\omega\tau = k^2T_1/(k^2+\omega^2)$ and $R\sin\omega\tau = k\omega T_1/(k^2+\omega^2)$ since these are the coefficients of $\cos\omega t$ and $\sin\omega t$, respectively. By squaring and adding we find $R^2 = k^2T_1^2/(k^2+\omega^2)$ and by dividing we find $\tan\omega\tau = \omega/k$.

19.(a) The required differential equation is $dQ/dt = kr + P - Q(t)r/V$, since kr is the rate of water pollutant entering the lake, P is the rate of pollutant entering directly and $Q(t)r/V$ is the rate at which the pollutant leaves the lake. The initial condition is $Q(0) = Vc_0$. Since $c = Q(t)/V$, the initial value problem may be rewritten as $Vc'(t) = kr + P - rc$, $c(0) = c_0$, which has the solution $c(t) = k + P/r + (c_0 - k - P/r)e^{-rt/V}$. It is easy to see that $\lim_{t\to\infty} c(t) = k + P/r$.

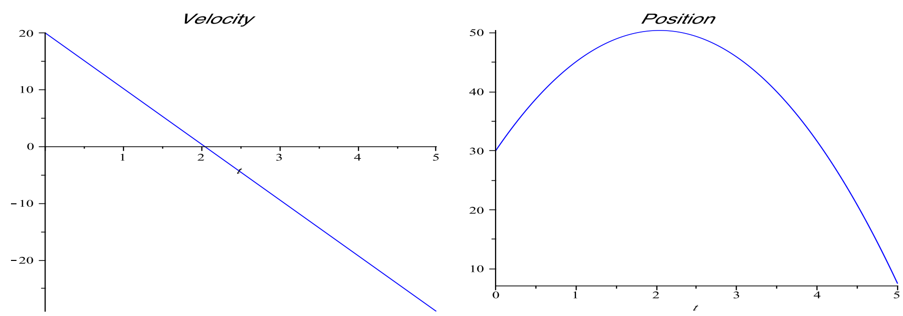
(b) $c(t) = c_0\, e^{-rt/V}$. The reduction times are $T_{50} = V\ln 2/r$ and $T_{10} = V\ln 10/r$.

(c) The reduction times are $T_S = (12,200)\ln 10/65.2 = 430.85$ years; for Lake Michigan, $T_M = (4,900)\ln 10/158 = 71.4$ years; $T_E = (460)\ln 10/175 = 6.05$ years; and $T_O = (16,000)\ln 10/209 = 17.63$ years.

20.(a) If we measure x positively upward from the ground, then Eq.(4) of Section 1.1 becomes $mdv/dt = -mg$, since there is no air resistance. Thus the initial value problem for $v(t)$ is $dv/dt = -g$, $v(0) = 20$, which gives $v(t) = 20 - gt$. Since $dx/dt = v(t)$ we get $x(t) = 20t - (g/2)t^2 + c$. Then $x(0) = 30$ gives $c = 30$ and thus $x(t) = 20t - (g/2)t^2 + 30$. At the maximum height $v(t_m) = 0$ and thus $t_m = 20/9.8 = 2.04$ sec, which when substituted in the equation for $x(t)$ yields the maximum height.

(b) The ball hits the ground when $x(t) = 0$, solving this equation gives $t = 5.2$ sec.

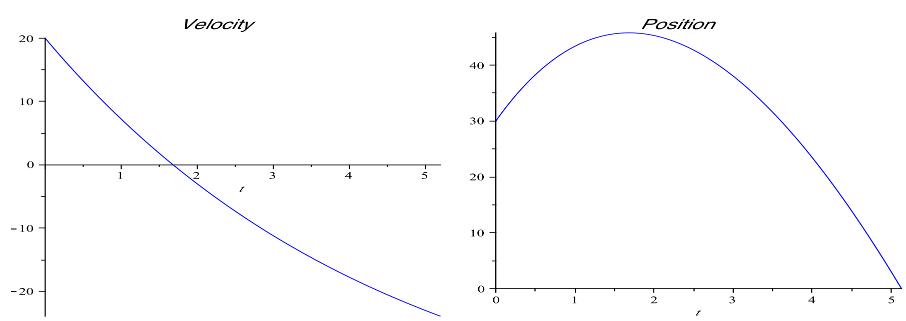
(c)



21.(a) The differential equation for the motion is $m\,dv/dt = -v/30 - mg$. Given the initial condition $v(0) = 20$ m/s , the solution is $v(t) = -44.1 + 64.1\,e^{-t/4.5}$. Setting $v(t_1) = 0$, the ball reaches the maximum height at $t_1 = 1.683$ s. Integrating $v(t)$, the position is given by $x(t) = 318.45 - 44.1\,t - 288.45\,e^{-t/4.5}$. Hence the maximum height is $x(t_1) = 45.78$ m.

(b) Setting $x(t_2) = 0$, the ball hits the ground at $t_2 = 5.128$ s.

(c)



23.(a) Measure the positive direction of motion downward. Based on Newton's second law, the equation of motion is given by

$$m\frac{dv}{dt} = \begin{cases} -0.75\,v + mg, & 0 < t < 10 \\ -12\,v + mg, & t > 10 \end{cases}.$$

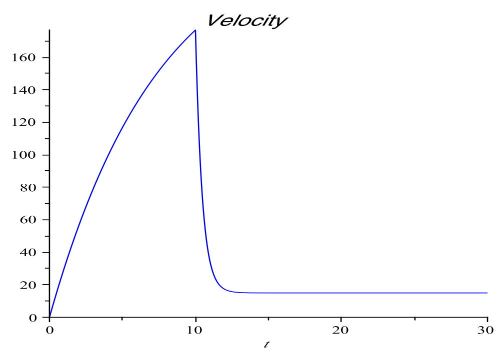
Note that gravity acts in the positive direction, and the drag force is resistive. During the first ten seconds of fall, the initial value problem is $dv/dt = -v/7.5 + 32$, with initial velocity $v(0) = 0$ ft/s. This differential equation is separable and linear, with solution $v(t) = 240(1 - e^{-t/7.5})$. Hence $v(10) = 176.7$ ft/s.

(b) Integrating the velocity, with $x(t) = 0$, the distance fallen is given by $x(t) = 240\,t + 1800\,e^{-t/7.5} - 1800$. Hence $x(10) = 1074.5$ ft.

(c) For computational purposes, reset time to $t = 0$. For the remainder of the motion, the initial value problem is $dv/dt = -32v/15 + 32$, with specified initial velocity $v(0) = 176.7$ ft/s. The solution is given by $v(t) = 15 + 161.7e^{-32\,t/15}$. As $t \to \infty$, $v(t) \to v_L = 15$ ft/s.

(d) Integrating the velocity, with $x(0) = 1074.5$, the distance fallen after the parachute is open is given by $x(t) = 15t - 75.8e^{-32t/15} + 1150.3$. To find the duration of the second part of the motion, estimate the root of the transcendental equation $15T - 75.8e^{-32T/15} + 1150.3 = 5000$. The result is $T = 256.6$ s.

(e)



26.(a) As in Problem 21, $mdv/dt = -mg - kv$, $v(0) = v_0$.

(b) From part (a), $v(t) = -mg/k + (v_0 + mg/k)e^{-kt/m}$. As $k \to 0$, this has the indeterminate form of $-\infty + \infty$. Thus rewrite $v(t)$ as

$$v(t) = (-mg + (v_0 k + mg)e^{-kt/m})/k,$$

which has the indeterminate form $0/0$ as $k \to 0$. Using L'Hospital's rule,

$$\lim_{k\to 0} \frac{-mg + (k\,v_0 + mg)e^{-kt/m}}{k} = \lim_{k\to 0}[v_0 e^{-kt/m} - \frac{t}{m}(k\,v_0 + mg)e^{-kt/m}] = v_0 - gt.$$

(c)

$$\lim_{m\to 0}\left[-\frac{mg}{k} + (\frac{mg}{k} + v_0)e^{-kt/m}\right] = 0,$$

since $\lim_{m\to 0} e^{-kt/m} = 0$.

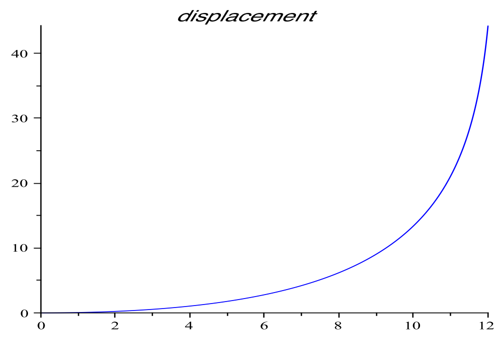
27.(a) The equation of motion is $m(dv/dt) = w - R - B$ which, in this problem, is $(4/3)\pi a^3\rho(dv/dt) = (4/3)\pi a^3\rho g - 6\pi\mu a v - (4/3)\pi a^3\rho' g$. The limiting velocity occurs when $dv/dt = 0$.

(b) Since the droplet is motionless, $v = dv/dt = 0$, we have the equation of motion $0 = (4/3)\pi a^3\rho g - Ee - (4/3)\pi a^3\rho' g$, where ρ is the density of the oil and ρ' is the density of air. Solving for e yields the answer.

28.(a) In terms of displacement, the differential equation is $mvdv/dx = -kv + mg$. This follows from the chain rule: $dv/dt = (dv/dx)(dx/dt) = vdv/dx$. The differential equation is separable, $x(0) = 0$, so

$$x(v) = -\frac{mv}{k} - \frac{m^2 g}{k^2} \ln\left|\frac{mg - k\,v}{mg}\right|.$$

The inverse exists, since both x and v are monotone increasing. In terms of the given parameters, $x(v) = -1.25\,v - 15.32\,\ln|0.0816\,v - 1|$.



(b) $x(10) = 13.45$ meters.

(c) In part (a), set $v = 10$ m/s and $x = 30$ meters. Solving numerically, the required value is $k = 0.239$.

29.(a) Let x represent the height above the earth's surface. The equation of motion is given by $mdv/dt = -GMm/(R+x)^2$, in which G is the universal gravitational constant. The symbols M and R are the mass and radius of the Earth, respectively. By the chain rule,

$$mv\frac{dv}{dx} = -G\frac{Mm}{(R+x)^2}.$$

This equation is separable, with $vdv = -GM(R+x)^{-2}dx$. Integrating both sides, and invoking the initial condition $v(0) = \sqrt{2gR}$, the solution is

$$v^2 = 2GM(R+x)^{-1} + 2gR - 2GM/R.$$

From elementary physics, it follows that $g = GM/R^2$, and $v(x) = \sqrt{2g}\,(R/\sqrt{R+x})$. (Note that $g = 78,545$ mi/hr^2.)

(b) We now consider $dx/dt = \sqrt{2g}\,(R/\sqrt{R+x})$. This equation is also separable, with $\sqrt{R+x}\,dx = \sqrt{2g}\,Rdt$. By definition of the variable x, the initial condition is $x(0) = 0$. Integrating both sides, we obtain

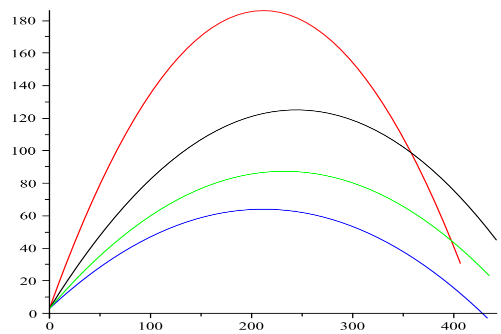
$$x(t) = (\frac{3}{2}(\sqrt{2g}\,R\,t + \frac{2}{3}R^{3/2}))^{2/3} - R.$$

Setting the distance $x(T) + R = 244,000$, and solving for T, the duration of such a flight would be $T \approx 50.6$ hours.

30.(a) We obtain these by solving the given differential equations with the initial conditions $v(0) = u\cos A$ and $w(0) = u\sin A$.

(b) From part (a) $dx/dt = v = u\cos A$ and hence $x(t) = tu\cos A + d_1$. Since $x(0) = 0$, we have $d_1 = 0$ and $x(t) = tu\cos A$. Likewise, $dy/dt = w = -gt + u\sin A$ and therefore $y(t) = -gt^2/2 + tu\sin A + d_2$. Since $y(0) = h$ we have $d_2 = h$ and $y(t) = -gt^2/2 + tu\sin A + h$.

(c)



(d) Let t_w be the time the ball reaches the wall. Then $x(t_w)) = L = t_w u\cos A$ and thus $t_w = L/(u\cos A)$. For the ball to clear the wall $y(t_w) \geq H$ and thus (setting $t_w = L/(u\cos A)$, $g = 32$ and $h = 3$ in y) we get $-16L^2/(u^2\cos^2 A) + L\tan A + 3 \geq H$.

(e) Setting $L = 350$ and $H = 10$ we get $-161.98/\cos^2 A + 350\tan A \geq 7$ or $7\cos^2 A - 350\cos A\sin A + 161.98 \leq 0$. This can be solved numerically or by plotting the left side as a function of A and finding where the zero crossings are.

(f) Setting $L = 350$ and $H = 10$ in the answer to part (d) yields the equation $-16(350)^2/(u^2\cos^2 A) + 350\tan A = 7$, where we have chosen the equality sign since we want to just clear the wall. Solving for u^2, we obtain that in this case $u^2 = 1,960,000/(175\sin 2A - 7\cos^2 A)$. Now u will have a minimum when the denominator has a maximum. Thus $350\cos 2A + 7\sin 2A = 0$, or $\tan 2A = -50$, which yields $A = 0.7954$ rad and $u = 106.89$ ft/sec.

2.4

1. If the equation is written in the form of Eq.(1), then $p(t) = \ln t/(t-3)$ and $g(t) = 2t/(t-3)$. These are defined and continuous on the intervals $(0, 3)$ and $(3, \infty)$, but since the initial point is $t = 1$, the solution will be continuous on $0 < t < 3$.

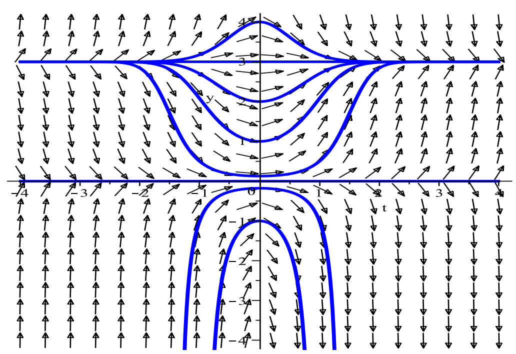
4. $p(t) = 2t/(4-t^2)$ and $g(t) = 3t^2/(4-t^2)$, which have discontinuities at $t = \pm 2$. Since $t_0 = -3$, the solution will be continuous on $-\infty < t < -2$.

8. Theorem 2.4.2 guarantees a unique solution to the differential equation through any point (t_0, y_0) such that $t_0^2 + y_0^2 < 1$ since $\partial f/\partial y = -y/(1-t^2-y^2)^{1/2}$ is defined and continuous only for $1 - t^2 - y^2 > 0$. Note also that $f = (1-t^2-y^2)^{1/2}$ is defined and continuous in this region as well as on the boundary $t^2 + y^2 = 1$. The boundary can't be included in the final region due to the discontinuity of $\partial f/\partial y$ there.

11. In this case $f = (1+t^2)/(y(3-y))$, and then $\partial f/\partial y = (1+t^2)/(y(3-y)^2) - (1+t^2)/(y^2(3-y))$, which are both continuous everywhere except for $y = 0$ and $y = 3$.

13. The differential equation can be written as $ydy = -4tdt$, so $y^2/2 = -2t^2 + c$, or $y^2 = c - 4t^2$. The initial condition then yields $y_0^2 = c$, so that $y^2 = y_0^2 - 4t^2$ or $y = \pm\sqrt{y_0^2 - 4t^2}$, which is defined for $4t^2 < y_0^2$ or $|t| < |y_0|/2$. Note that $y_0 \neq 0$ since Theorem 2.4.2 does not hold there.

17.



From the direction field (or the given differential equation) it is noted that for $t > 0$ and $y < 0$ that $y' < 0$, so $y \to -\infty$ for $y_0 < 0$. Likewise, for $0 < y_0 < 3$, $y' > 0$ and $y' \to 0$ as $y \to 3$, so $y \to 3$ for $0 < y_0 < 3$ and for $y_0 > 3$, $y' < 0$ and again $y' \to 0$ as $y \to 3$, so $y \to 3$ for $y_0 > 3$. For $y_0 = 3$, $y' = 0$ and $y = 3$ for all t and for $y_0 = 0$, $y' = 0$ and $y = 0$ for all t.

22.(a) For $y_1 = 1 - t$, $y_1' = -1$, so substitution into the differential equation gives $-1 = (-t + \sqrt{t^2 + 4(1-t)})/2 = (-t + \sqrt{(t-2)^2})/2 = (-t + |t-2|)/2$. By the definition of the absolute value, the right side is -1 if $t - 2 \geq 0$. Setting $t = 2$ in y_1 we get $y_1(2) = -1$, as required by the initial condition. For $y_2 = -t^2/4$, $y_2' = -t/2$ so substitution into the differential equation yields $-t/2 = (-t + \sqrt{t^2 + 4(-t^2/4)})/2 = -t/2$ which is valid for all t values. Also, $y_2(2) = -1$.

(b) By Theorem 2.4.2 we are guaranteed a unique solution only where $f(t,y) = (-t + \sqrt{t^2 + 4y})/2$ and $f_y(t,y) = 1/\sqrt{t^2 + 4y}$ are continuous. In this case the initial point $(2, -1)$ lies in the region $t^2 + 4y \leq 0$, so $\partial f/\partial y$ is not continuous and hence the theorem is not applicable and there is no contradiction.

(c) For $y = ct + c^2$ follow the steps of part (a). If $y = y_2(t)$ then we must have $ct + c^2 = -t^2/4$ for all t, which is not possible since c is a constant.

23.(a) $\phi(t) = e^{2t}$ gives $\phi'(t) = 2e^{2t}$ so $\phi' - 2\phi = 0$. $\phi(t) = ce^{2t}$ gives $\phi'(t) = 2ce^{2t}$, so $\phi' - 2\phi = 0$.

(b) $\phi(t) = t^{-1}$ gives $\phi'(t) = -t^{-2}$ so $\phi' + \phi^2 = 0$. $\phi(t) = ct^{-1}$ gives $\phi'(t) = -ct^{-2}$, so $\phi' + \phi^2 \neq 0$ unless $c = 0$ or $c = 1$.

25. $(y_1(t) + y_2(t))' + p(t)(y_1(t) + y_2(t)) = y_1'(t) + p(t)y_1(t) + y_2'(t) + p(t)y_2(t) = 0 + g(t)$.

27.(a) For $n = 1$, we have $y' + (p(t) - q(t))y = 0$, which is linear. Thus Eq.(3) gives $y(t) = c\mu^{-1}(t) = ce^{-\int (p(t)-q(t))\,dt}$, since $g(t) = 0$.

(b) Let $v = y^{1-n}$, then $dv/dt = (1-n)y^{-n}dy/dt$, so $dy/dt = (1/(1-n))y^n dv/dt$, for $n \neq 1$. Substituting into the differential equation yields $(1/(1-n))y^n dv/dt + p(t)y = q(t)y^n$, or $v' + (1-n)p(t)y^{1-n} = (1-n)q(t)$, which is $v' + (1-n)p(t)v = (1-n)q(t)$, which is a linear differential equation for v.

28. Here $n = 3$, so $v = y^{-2}$ and $dv/dt = -2y^{-3}dy/dt$ or $dy/dt = -(1/2)y^3 dv/dt$. Substituting this into the equation gives $-(1/2)y^3 dv/dt + (2/t)y = (1/t^2)y^3$. Simplifying and using $y^{-2} = v$ then gives the linear differential equation $v' - (4/t)v = -(2/t^2)$. Thus $\mu(t) = 1/t^4$ and $v(t) = ct^4 + 2/(5t) = (2 + 5ct^5)/(5t)$. Solving for y gives $y = \pm\sqrt{5t/(2 + 5ct^5)}$.

29. Here $n = 2$, so $v = y^{-1}$ and $dv/dt = -y^{-2}dy/dt$. Thus the differential equation becomes $-y^{-2}dv/dt - ry = -ky^2$ or $dv/dt + rv = k$. Hence $\mu(t) = e^{rt}$ and $v = (k/r) + ce^{-rt}$. Setting $v = 1/y$ then yields the solution.

32. Since $g(t)$ is continuous on the interval $0 \le t \le 1$ and hence we solve the initial value problem $y_1' + 2y_1 = 1,\ y_1(0) = 0$ on that interval to obtain $y_1 = 1/2 - (1/2)e^{-2t}$, $0 \le t \le 1$. For $1 < t$, $g(t) = 0$; and hence we solve $y_2' + 2y_2 = 0$ to obtain $y_2 = ce^{-2t}$, $1 < t$. The solution y of the original initial value problem must be continuous at $t = 1$ (since its derivative must exist) and hence we need c in y_2 so y_2 at 1 has the same value as y_1 at 1. Thus $ce^{-2} = 1/2 - e^{-2}/2$ or $c = (1/2)(e^2 - 1)$ and we obtain

$$y(t) = \begin{cases} \frac{1}{2} - \frac{1}{2}e^{-2t}, & 0 \le t \le 1 \\ \frac{1}{2}(e^2 - 1)e^{-2t}, & t > 1 \end{cases}.$$

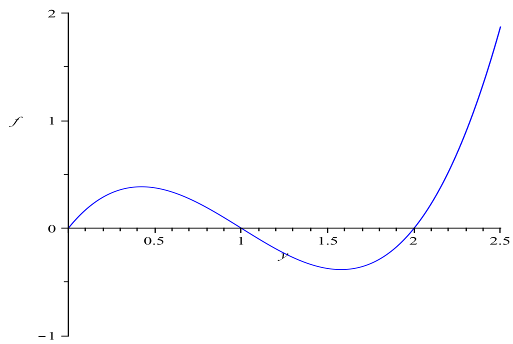
and

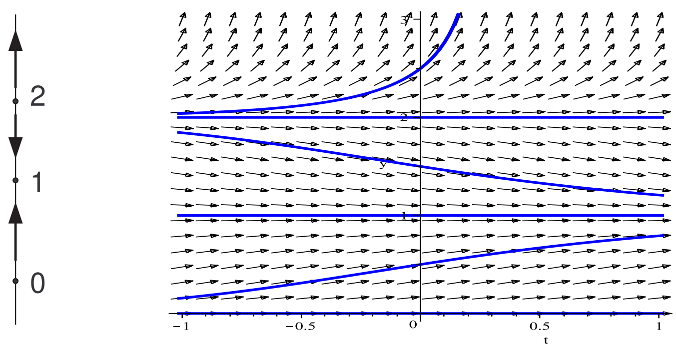
$$y'(t) = \begin{cases} e^{-2t}, & 0 < t < 1 \\ (1 - e^2)e^{-2t}, & t > 1 \end{cases}.$$

Evaluating the two parts of y' at $t_0 = 1$ we see that they are different, and hence y' is not continuous at $t_0 = 1$.

2.5

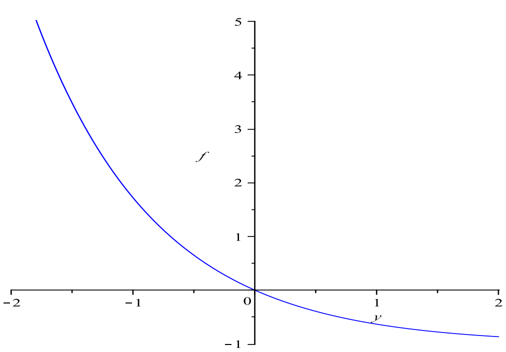
3.

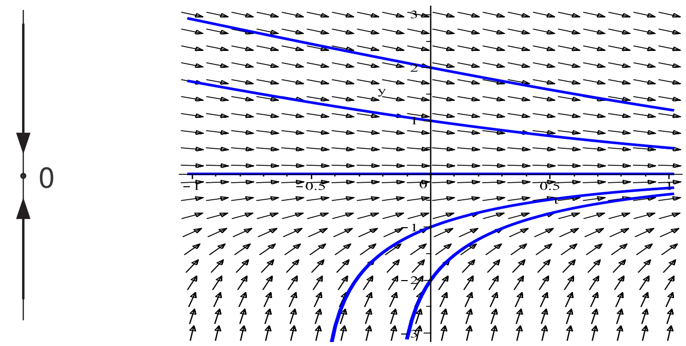




From the graph, $y' < 0$ when $1 < y < 2$ and $y' > 0$ when $0 < y < 1$ or $y > 2$, so the equilibrium solutions $y = 0$ and $y = 2$ are unstable, the equilibrium solution $y = 1$ is asymptotically stable.

5.

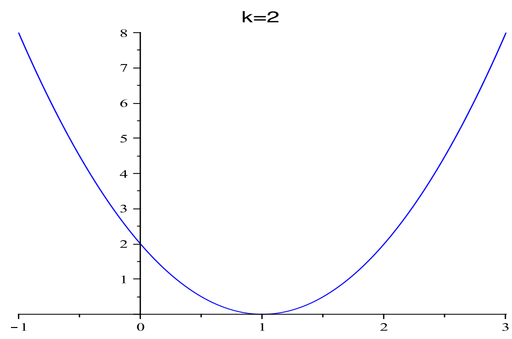




The only equilibrium point is $y^* = 0$, and $y' > 0$ when $y < 0$, $y' < 0$ when $y > 0$, hence the equilibrium solution $y = 0$ is asymptotically stable.

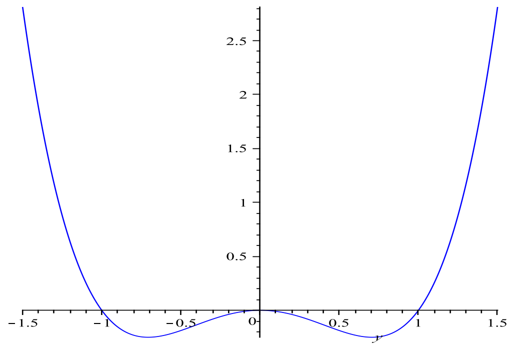
7.(a) $f(y) = 0$ only when $y = 1$. Therefore, $y^* = 1$ is the only critical point.

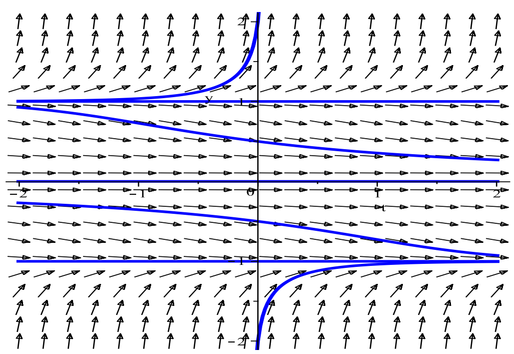
(b)



(c) Separate variables to get $dy/(1-y)^2 = kdt$. Integration yields $1/(1-y) = kt + c$, or $y = (kt + c - 1)/(kt + c)$. Setting $t = 0$ and $y(0) = y_0$ yields $y_0 = (c-1)/c$ or $c = 1/(1-y_0)$. Hence $y(t) = [y_0 + (1-y_0)kt]/[1 + (1-y_0)kt]$. If $y_0 < 1$, then $y \to 1$ as $t \to \infty$. If $y_0 > 1$, then the denominator will go to zero at some finite time $T = 1/(y_0 - 1)$. Therefore, the solution will go towards infinity at that time T.

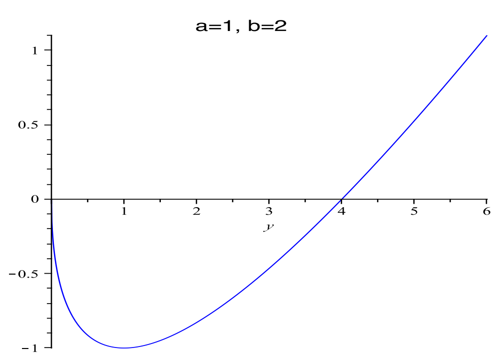
9.

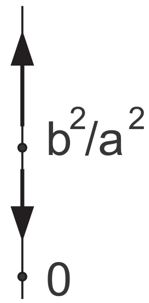


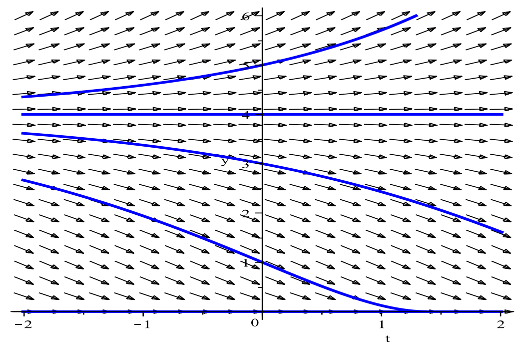


The critical points are $y = 0, \pm 1$. We have $y' > 0$ for $|y| > 1$ while $y' < 0$ for $|y| < 1$. Thus the equilibrium solution $y = -1$ is asymptotically stable, $y = 0$ is semistable and $y = 1$ is unstable.

11.



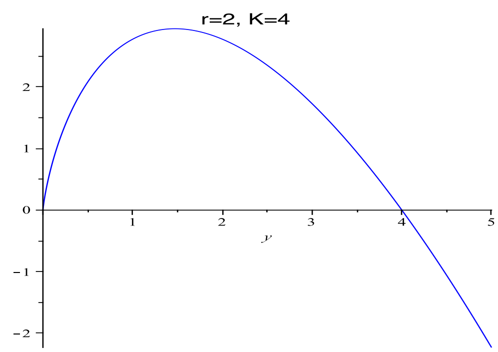




$y = b^2/a^2$ and $y = 0$ are the only critical points. For $0 < y < b^2/a^2$, $y' < 0$. For $y > b^2/a^2$, $y' > 0$. Thus the equilibrium solution $y = 0$ is asymptotically stable, the equilibrium solution $y = b^2/a^2$ is unstable.

14. If $f'(y_1) < 0$ then the slope of f is negative at y_1 and thus $f(y) > 0$ for $y < y_1$ and $f(y) < 0$ for $y > y_1$ since $f(y_1) = 0$. Hence y_1 is an asymptotically stable critical point. A similar argument will yield the result for $f'(y_1) > 0$.

16.(a)



The critical points occur at $y^* = 0, K$. Since $f'(0) > 0$, $y^* = 0$ is unstable. Since $f'(K) < 0$, $y^* = K$ is asymptotically stable.

(b) We calculate y''. Using the chain rule, we see that

$$y'' = ry'\left[\ln\left(\frac{K}{y}\right) - 1\right].$$

We see that $y'' = 0$ when $y' = 0$ (meaning $y = 0, K$) or when $\ln(K/y) - 1 = 0$, meaning $y = K/e$. Looking at the sign of y'' in the intervals $0 < y < K/e$ and $K/e < y < K$, we conclude that y is concave up in the interval $0 < y < K/e$ and concave down in the interval $K/e < y < K$.

(c) $\ln(K/y)$ is very large for small values of y and thus $ry\ln(K/y) > ry(1 - y/K)$ for small y. Since $\ln(K/y)$ and $(1 - y/K)$ are both strictly decreasing functions of y and since $\ln(K/y) = (1 - y/K)$ only for $y = K$, we may conclude that $y' = ry\ln(K/y)$ is never less than $y' = ry(1 - y/K)$.

17.(a) Consider the change of variable $u = \ln(y/K)$. Differentiating both sides with respect to t, $u' = y'/y$. Substitution into the Gompertz equation yields $u' = -ru$, with solution $u = u_0 e^{-rt}$. It follows that $\ln(y/K) = \ln(y_0/K)e^{-rt}$. This implies that $y/K = e^{\ln(y_0/K)e^{-rt}}$.

(b) Given $K = 80.5 \times 10^6$, $y_0/K = 0.25$ and $r = 0.71$ per year, $y(2) = 57.58 \times 10^6$.

(c) Solving for t,

$$t = -\frac{1}{r} \ln \left[\frac{\ln(y/K)}{\ln(y_0/K)} \right].$$

Setting $y(\tau) = 0.75K$, the corresponding time is $\tau \approx 2.21$ years.

18.(a) The differential equation is $dV/dt = k - \alpha\pi r^2$. The volume of a cone of height L and radius r is given by $V = \pi r^2 L/3$ where $L = hr/a$ from symmetry. Solving for r yields the desired equation $dV/dt = k - \alpha\pi(3a/\pi h)^{2/3} V^{2/3}$.

(b) The equilibrium is given by the equation $k = \alpha\pi r^2$, which yields $r = \sqrt{k/\alpha\pi}$ and then $L = h\sqrt{k/\alpha\pi}/a$. By checking the graph of V' we obtain that this is an asymptotically stable equilibrium point.

(c) The equilibrium height must be less than h, or $\sqrt{k/\alpha\pi}/a < 1$.

20.(a) If $E < r$, then the equilibrium points are given by $0 = r(1 - y/K)y - Ey = y(r - ry/K - E)$, which means that either $y = 0$ or $y = (r - E)K/r = (1 - E/r)K > 0$.

(b) $f'(y) = r - E - 2ry/K$, so $f'(0) = r - E >$ and 0 is an unstable equilibrium, while $f'((1 - E/r)K) = E - r < 0$ and $(1 - E/r)K$ is an asymptotically stable equilibrium.

(c) $Y = E(1 - E/r)K$.

(d) We have to solve $0 = dY/dE = K - 2EK/r$ to get $E = r/2$, and then $Y_m = rK/4$.

21.(a) Setting $dy/dt = 0$ the quadratic formula yields the roots

$$y_{1,2} = \frac{r \pm \sqrt{r^2 - 4rh/K}}{2r/K} = \frac{K}{2}\left(1 \pm \sqrt{1 - \frac{4h}{rK}}\right),$$

which are real when $h < rK/4$. ($y_1 < y_2$ because of the minus sign in front of the square root.)

(b) The graph of the right side of the differential equation is a downward opening parabola, which implies that y_1 is unstable and y_2 is asymptotically stable. We can also use the derivative test of Problem 14.

(c) The graph of $f(y)$ is a downward opening parabola intersecting the horizontal axis at y_1 and y_2, so we know that between y_1 and y_2 the value of $y' = f(y)$ is positive, which implies that if $y_1 < y_0 < y_2$, then the solution is increasing towards y_2, and when $y_2 < y_0$, the solution is decreasing towards y_2 (because $y' = f(y)$ is negative there). Also, when $y_0 < y_1$, then $y' < 0$, so the solution will decrease and reach 0 in finite time.

(d) If $h > rK/4$ there are no critical points (see part (a)) and $dy/dt < 0$ for all t.

(e) We can see from part (a) that when $h = rK/4$, then $y_1 = y_2$. The graph of $f(y)$ intersects the horizontal axis at a single point of tangency in this case, and $y' = f(y)$ is negative for any other y value, giving the semistability result.

24.(a) Letting $' = d/dt$, we obtain that the derivative is $z' = (nx' - xn')/n^2 = (-\beta nx - \mu nx + \nu\beta x^2 + \mu nx)/n^2 = -\beta x/n + \nu\beta x^2/n^2 = -\beta z + \nu\beta z^2 = -\beta z(1 - \nu z)$.

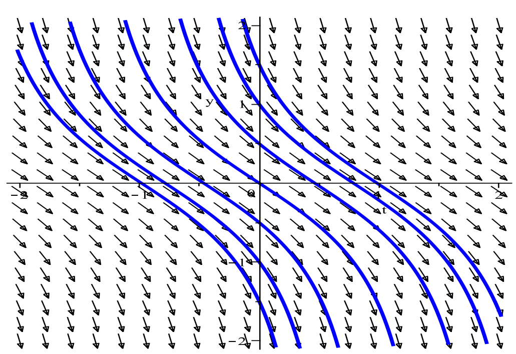
(b) First, we rewrite the equation as $z' + \beta z = \beta\nu z^2$. This is a Bernoulli equation with $n = 2$. Let $w = z^{1-n} = z^{-1}$. Then, our equation can be written as $w' - \beta w = -\beta\nu$. This is a linear equation with solution $w = \nu + ce^{\beta t}$. Then, using the fact that $z = 1/w$, we see that $z = 1/(\nu + ce^{\beta t})$. Finally, the initial condition $z(0) = 1$ implies $c = 1 - \nu$. Therefore, $z(t) = 1/(\nu + (1 - \nu)e^{\beta t})$.

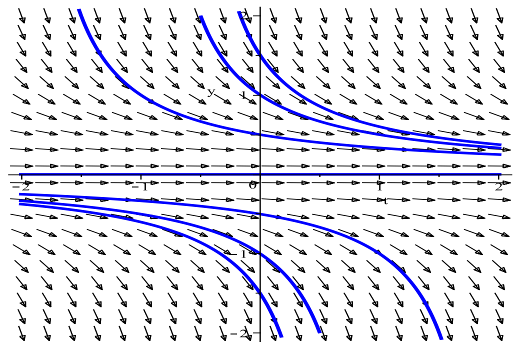
(c) Evaluating $z(20)$ for $\nu = \beta = 1/8$, we conclude that $z(20) = 0.0927$.

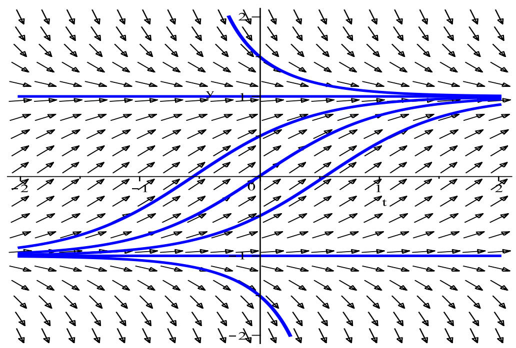
25.(a) The critical points occur when $a - y^2 = 0$. If $a < 0$, there are no critical points. If $a = 0$, then $y^* = 0$ is the only critical point. If $a > 0$, then $y^* = \pm\sqrt{a}$ are the two critical points.

(b) We note that $f'(y) = -2y$. Therefore, $f'(\sqrt{a}) < 0$ which implies that $\sqrt{a}$ is asymptotically stable; $f'(-\sqrt{a}) > 0$ which implies $-\sqrt{a}$ is unstable; the behavior of f' around $y^* = 0$ implies that $y^* = 0$ is semistable.

(c) Below, we graph solutions in the case $a = -1$, $a = 0$ and $a = 1$ respectively.







28. (a) Since the critical points are $x^* = p, q$, we will look at their stability. Since $f'(x) = -\alpha q - \alpha p + 2\alpha x^2$, we see that $f'(p) = \alpha(2p^2 - q - p)$ and $f'(q) = \alpha(2q^2 - q - p)$. Now if $p > q$, then $-p < -q$, and, therefore, $f'(q) = \alpha(2q^2 - q - p) < \alpha(2q^2 - 2q) = 2\alpha q(q-1) < 0$ since $0 < q < 1$. Therefore, if $p > q$, $f'(q) < 0$, and, therefore, $x^* = q$ is asymptotically stable. Similarly, if $p < q$, then $x^* = p$ is asymptotically stable, and therefore, we can conclude that $x(t) \to \min\{p, q\}$ as $t \to \infty$.

The equation is separable. It can be solved by using partial fractions as follows. We can rewrite the equation as

$$\left(\frac{1/(q-p)}{p-x} + \frac{1/(p-q)}{q-x}\right) dx = \alpha dt,$$

which implies

$$\ln\left|\frac{p-x}{q-x}\right| = (p-q)\alpha t + C.$$

The initial condition $x_0 = 0$ implies $C = \ln|p/q|$, and, therefore,

$$\ln\left|\frac{q(p-x)}{p(q-x)}\right| = (p-q)\alpha t.$$

Applying the exponential function and simplifying, we conclude that

$$x(t) = \frac{pq(e^{(p-q)\alpha t} - 1)}{pe^{(p-q)\alpha t} - q} = \frac{pq(e^{\alpha(q-p)t} - 1)}{qe^{\alpha(q-p)t} - p}.$$

(b) In this case, $x^* = p$ is the only critical point. Since $f(x) = \alpha(p-x)^2$ is concave up, we conclude that $x^* = p$ is semistable. Further, if $x_0 = 0$, we can conclude that $x \to p$ as $t \to \infty$. This equation is separable. Its solution is given by $x(t) = p^2\alpha t/(p\alpha t + 1)$.

2.6

3. Here $M(x,y) = 3x^2 - 2xy + 2$ and $N(x,y) = 6y^2 - x^2 + 3$. Since $M_y = -2x = N_x$, the equation is exact. Since $\psi_x = M = 3x^2 - 2xy + 2$, to solve for ψ, we integrate M with respect to x. We conclude that $\psi = x^3 - x^2y + 2x + h(y)$. Then $\psi_y = -x^2 + h'(y) = N = 6y^2 - x^2 + 3$ implies $h'(y) = 6y^2 + 3$. Therefore, $h(y) = 2y^3 + 3y$ and $\psi(x,y) = x^3 - x^2y + 2x + 2y^3 + 3y = c$.

5. Here $M(x,y) = ax + by$ and $N(x,y) = bx + cy$. Since $M_y = b = N_x$, the equation is exact. Since $\psi_x = M = ax + by$, to solve for ψ, we integrate M with respect to x. We conclude that $\psi = ax^2/2 + bxy + h(y)$. Then $\psi_y = bx + h'(y) = N = bx + cy$ implies $h'(y) = cy$. Therefore, $h(y) = cy^2/2$ and $\psi(x,y) = ax^2 + 2bxy + cy^2 = k$.

7. Here $M(x,y) = e^x \sin y - 2y \sin x$ and $N(x,y) = e^x \cos y + 2\cos x$. Since $M_y = e^x \cos y - 2\sin x = N_x$, the equation is exact. Since $\psi_x = M = e^x \sin y - 2y\sin x$, to solve for ψ, we integrate M with respect to x. We conclude that $\psi = e^x \sin y + 2y\cos x + h(y)$. Then $\psi_y = e^x \cos y + 2\cos x + h'(y) = N = e^x \cos y + 2\cos x$ implies $h'(y) = 0$. Therefore, $h(y) = c$ and $\psi(x,y) = e^x \sin y + 2y\cos x = c$.

9. Here $M(x,y) = ye^{xy}\cos(2x) - 2e^{xy}\sin(2x) + 2x$ and $N(x,y) = xe^{xy}\cos(2x) - 3$. Since $M_y = e^{xy}\cos(2x) + xye^{xy}\cos(2x) - 2xe^{xy}\sin(2x) = N_x$, the equation is exact. If we try to find $\psi(x,y)$ by integrating $M(x,y)$ with respect to x we must integrate by parts. Instead we find $\psi(x,y)$ by integrating $N(x,y)$ with respect to y to obtain $\psi(x,y) = e^{xy}\cos(2x) - 3y + g(x)$. Then we find $g(x)$ by differentiating $\psi(x,y)$ with respect to x and setting it equal to $M(x,y)$, resulting in $g'(x) = 2x$ or $g(x) = x^2$. As before, the implicit solution is $\psi(x,y) = e^{xy}\cos(2x) + x^2 - 3y = c$.

12. Here $M(x,y) = x/(x^2+y^2)^{3/2}$ and $N(x,y) = y/(x^2+y^2)^{3/2}$. Since $M_y = N_x$, the equation is exact. Since $\psi_x = M = x/(x^2+y^2)^{3/2}$, to solve for ψ, we integrate M with respect to x. We conclude that $\psi = -1/(x^2+y^2)^{1/2} + h(y)$. Then $\psi_y = y/(x^2+y^2)^{3/2} + h'(y) = N = y/(x^2+y^2)^{3/2}$ implies $h'(y) = 0$. Therefore, $h(y) = 0$ and $\psi(x,y) = -1/(x^2+y^2)^{1/2} = c$ or $x^2 + y^2 = k$. We can observe that as long as $x^2 + y^2 \neq 0$, we can simplify the equation by multiplying both sides by $(x^2+y^2)^{3/2}$. This gives the (simpler) exact equation $xdx + ydy = 0$, whose solution is the same as the above.

14. Here $M(x,y) = 9x^2 + y - 1$ and $N(x,y) = -4y + x$. Therefore, $M_y = N_x = 1$ which implies that the equation is exact. Integrating M with respect to x, we conclude that $\psi = 3x^3 + xy - x + h(y)$. Then $\psi_y = x + h'(y) = N = -4y + x$ implies $h'(y) = -4y$. Therefore, $h(y) = -2y^2$ and we get $\psi = 3x^3 + xy - x - 2y^2 = c$. The

initial condition $y(1) = 0$ implies $c = 2$. Therefore, $3x^3 + xy - x - 2y^2 = 2$. Solving for y using the quadratic formula, we get $y = [x - (24x^3 + x^2 - 8x - 16)^{1/2}]/4$. Using a numerical process the square root term is positive for $x > 0.9846$.

15. Here $M(x,y) = xy^2 + bx^2y$ and $N(x,y) = x^3 + x^2y$. Therefore, $M_y = 2xy + bx^2$ and $N_x = 3x^2 + 2xy$. In order for the equation to be exact, we need $b = 3$. Taking this value for b, we integrating M with respect to x. We conclude that $\psi = x^2y^2/2 + x^3y + h(y)$. Then $\psi_y = x^2y + x^3 + h'(y) = N = x^3 + x^2y$ implies $h'(y) = 0$. Therefore, $h(y) = c$ and $\psi(x,y) = x^2y^2/2 + x^3y = c$. That is, the solution is given implicitly as $x^2y^2 + 2x^3y = k$.

19. Here $M(x,y) = x^2y^3$ and $N(x,y) = x + xy^2$. Therefore, $M_y = 3x^2y^2$ and $N_x = 1 + y^2$. We see that the equation is not exact. Now, multiplying the equation by $\mu(x,y) = 1/xy^3$, the equation becomes $xdx + (1+y^2)/y^3 dy = 0$. Now we see that for this equation $M = x$ and $N = (1+y^2)/y^3$. Therefore, $M_y = 0 = N_x$. Integrating M with respect to x, we see that $\psi = x^2/2 + h(y)$. Further, $\psi_y = h'(y) = N = (1+y^2)/y^3 = 1/y^3 + 1/y$. Therefore, $h(y) = -1/2y^2 + \ln y$ and we conclude that the solution of the equation is given implicitly by $x^2 - 1/y^2 + 2\ln y = c$ and $y = 0$.

22. We see that $M_y = (x+2)\cos y$ while $N_x = \cos y$. Therefore, $M_y \neq N_x$. However, multiplying the equation by the given integrating factor $\mu(x,y) = xe^x$, this becomes $(x^2 + 2x)e^x \sin y dx + x^2e^x \cos y dy = 0$. Now we see that for this equation $M_y = (x^2 + 2x)e^x \cos y = N_x$. To solve this exact equation it is easiest to integrate (the new) N with respect to y to get $\psi(x,y) = x^2e^x \sin y + g(x)$. Finding ψ_x and setting it equal to (the new) M yields $g'(x) = 0$, which implies that the solution of the equation is given implicitly by $x^2e^x \sin y = c$.

23. Suppose μ is an integrating factor which will make the equation exact. Then multiplying the equation by μ, we have $\mu M dx + \mu N dy = 0$. Then we need $(\mu M)_y = (\mu N)_x$. That is, we need $\mu_y M + \mu M_y = \mu_x N + \mu N_x$. Then we rewrite the equation as $\mu(N_x - M_y) = \mu_y M - \mu_x N$. Suppose μ does not depend on x. Then $\mu_x = 0$. Therefore, $\mu(N_x - M_y) = \mu_y M$. Using the assumption that $(N_x - M_y)/M = Q(y)$, we can find an integrating factor μ by choosing μ which satisfies $\mu_y/\mu = Q$. We conclude that $\mu(y) = \exp \int Q(y)\, dy$ is an integrating factor of the differential equation.

25. Since $(M_y - N_x)/N = 3$ is a function of x only, we know that $\mu = e^{3x}$ is an integrating factor for this equation. Multiplying the equation by μ, we obtain the equation $e^{3x}(3x^2y + 2xy + y^3)dx + e^{3x}(x^2 + y^2)dy = 0$. Then $M_y = e^{3x}(3x^2 + 2x + 3y^2) = N_x$. Therefore, this new equation is exact. Integrating M with respect to x, we conclude that $\psi = (x^2y + y^3/3)e^{3x} + h(y)$. Then $\psi_y = (x^2 + y^2)e^{3x} + h'(y) = N = e^{3x}(x^2 + y^2)$. Therefore, $h'(y) = 0$ and we conclude that the solution is given implicitly by $(3x^2y + y^3)e^{3x} = c$.

26. Since $(M_y - N_x)/N = -1$ is a function of x only, we know that $\mu = e^{-x}$ is an integrating factor for this equation. Multiplying the equation by μ, we obtain the equation $(e^{-x} - e^x - ye^{-x})dx + e^{-x}dy = 0$. Then $M_y = -e^{-x} = N_x$. Therefore,

this new equation is exact. Integrating M with respect to x, we conclude that $\psi = -e^{-x} - e^{x} + ye^{-x} + h(y)$. Then $\psi_y = e^{-x} + h'(y) = N = e^{-x}$. Therefore, $h'(y) = 0$ and we conclude that the solution is given implicitly by $-e^{-x} - e^{x} + ye^{-x} = c$. Alternatively, we might recognize that $y' - y = e^{2x} - 1$ is a linear first order equation which can be solved as in Section 2.1.

27. Since $(N_x - M_y)/M = 1/y$ is a function of y only, we know by Problem 23 that $\mu(y) = e^{\int 1/y\, dy} = y$ is an integrating factor for this equation. Multiplying the equation by μ, we obtain $ydx + (x - y\sin y)dy = 0$. Then for this equation, $M_y = 1 = N_x$. Therefore, this new equation is exact. Integrating M with respect to x, we conclude that $\psi = xy + h(y)$. Then $\psi_y = x + h'(y) = N = x - y\sin y$. Therefore, $h'(y) = -y\sin y$ which implies that $h(y) = -\sin y + y\cos y$, and we conclude that the solution is given implicitly by $xy - \sin y + y\cos y = c$.

29. Since $(N_x - M_y)/M = \cot(y)$ is a function of y only, we know that $\mu(y) = e^{\int \cot(y)\, dy} = \sin(y)$ is an integrating factor for this equation. Multiplying the equation by μ, we obtain $e^x \sin y dx + (e^x \cos y + 2y)dy = 0$. Then for this equation, $M_y = N_x$. Therefore, this new equation is exact. Integrating M with respect to x, we conclude that $\psi = e^x \sin y + h(y)$. Then $\psi_y = e^x \cos y + h'(y) = N = e^x \cos y + 2y$. Therefore, $h'(y) = 2y$ which implies that $h(y) = y^2$, and we conclude that the solution is given implicitly by $e^x \sin y + y^2 = c$.

31. Since $(N_x - M_y)/(xM - yN) = 1/xy$ is a function of xy only, we know by Problem 24 that $\mu(xy) = e^{\int 1/xy\, dy} = xy$ is an integrating factor for this equation. Multiplying the equation by μ, we obtain $(3x^2y + 6x)dx + (x^3 + 3y^2)dy = 0$. Then for this equation, $M_y = N_x$. Therefore, this new equation is exact. Integrating M with respect to x, we conclude that $\psi = x^3y + 3x^2 + h(y)$. Then $\psi_y = x^3 + h'(y) = N = x^3 + 3y^2$. Therefore, $h'(y) = 3y^2$ which implies that $h(y) = y^3$, and we conclude that the solution is given implicitly by $x^3y + 3x^2 + y^3 = c$.

2.7

1. The Euler formula is $y_{n+1} = y_n + h(3 + t_n - y_n) = (1 - h)y_n + h(3 + t_n)$.

(a) 1.2, 1.39, 1.571, 1.7439

(b) 1.1975, 1.38549, 1.56491, 1.73658

(c) 1.19631, 1.38335, 1.56200, 1.73308

(d) The differential equation is linear with solution $y(t) = 2 + t - e^{-t}$. The values are 1.19516, 1.38127, 1.55918, 1.72968.

3. The Euler formula is $y_{n+1} = y_n + h(0.5 - t_n + 2y_n) = (1 + 2h)y_n + h(0.5 - t_n)$.

(a) 1.25, 1.54, 1.878, 2.2736

(b) 1.26, 1.5641, 1.92156, 2.34359

(c) 1.26551, 1.57746, 1.94586, 2.38287

(d) The differential equation is linear with solution $y(t) = 0.5t + e^{2t}$. The values are 1.2714, 1.59182, 1.97212, 2.42554.

4. The Euler formula is $y_{n+1} = y_n + h(3\cos(t_n) - 2y_n) = (1-2h)y_n + 3h\cos(t_n)$.

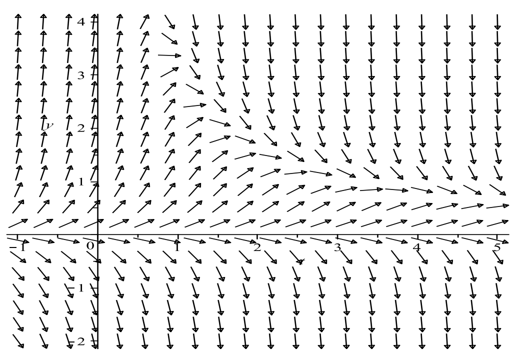
(a) 0.3, 0.538501, 0.724821, 0.866458

(b) 0.284813, 0.513339, 0.693451, 0.831571

(c) 0.277920, 0.501813, 0.678949, 0.815302

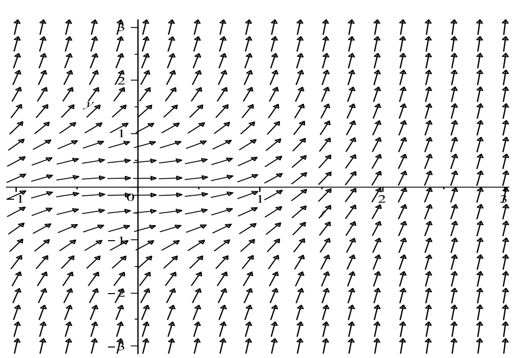
(d) The differential equation is linear with solution $y(t) = (6\cos(t) + 3\sin(t) - 6e^{-2t})/5$. The values are 0.271428, 0.490897, 0.665142, 0.799729.

6.



Solutions with $y(0) > 0$ appear to converge to a specific function, while solutions with $y(0) < 0$ decrease without bound. $y = 0$ is an equilibrium solution.

9.



All solutions seem to diverge.

13. The Euler formula is $y_{n+1} = y_n + h(4 - t_n y_n)/(1 + y_n^2)$ with $(t_0, y_0) = (0, -2)$.

(a) -1.48849, -0.412339, 1.04687, 1.43176, 1.54438, 1.51971

(b) -1.46909, -0.287883, 1.05351, 1.42003, 1.53000, 1.50549

(c) -1.45865, -0.217545, 1.05715, 1.41486, 1.52334, 1.49879

(d) -1.45212, -0.173376, 1.05941, 1.41197, 1.51949, 1.49490

15. The Euler formula is $y_{n+1} = y_n + h3t_n^2/(3y_n^2 - 4)$ with initial value $(t_0, y_0) = (1, 0)$.

(a) -0.166134, -0.410872, -0.804660, 4.15867

(b) -0.174652, -0.434238, -0.889140, -3.09810

(c) There are two factors that explain the large differences. From the differential equation, the slope of y, y', becomes very large for values of y near -1.155. Also, the slope changes sign at $y = -1.155$. Thus for part (a), $y(1.7) = y_7 = -1.178$, which is close to -1.155 and the slope y' here is large and positive, creating the large change in $y_8 = y(1.8)$. For part (b), $y(1.65) = -1.125$, resulting in a large negative slope, which yields $y(1.7) = -3.133$. The slope at this point is now positive and the remainder of the solutions grow to -3.098 for the approximation to $y(1.8)$.

16. The Euler formula is $y_{n+1} = y_n + h(t_n^2 + y_n^2)$ with $(t_0, y_0) = (0, 1)$. For the four step sizes given, the approximate values for $y(0.8)$ are 3.5078, 4.2013, 4.8004 and 5.3428. Thus, since these changes are still rather large, it is hard to give an estimate other than $y(0.8)$ is at least 5.3428. By using $h = 0.005$, 0.0025 and 0.001, we find further approximate values of $y(0.8)$ to be 5.576, 5.707 and 5.790. Thus a better estimate now is for $y(0.8)$ to be between 5.8 and 6. No reliable estimate is obtainable for $y(1)$, which is consistent with the direction field of Problem 9.

18.(a) See the direction field in Problem 8 above.

(b) The Euler formula is $y_{n+1} = y_n + h(-t_n y_n + 0.1y_n^3)$. For $y_0 < 2.37$, the solutions seem to converge, while the solutions seem to diverge if $y_0 > 2.38$. We conclude that $2.37 < \alpha_0 < 2.38$.

22. Using Eq.(8) we have $y_{n+1} = y_n + h(2y_n - 1) = (1 + 2h)y_n - h$. Setting $n + 1 = k$ (and hence $n = k - 1$) this becomes $y_k = (1 + 2h)y_{k-1} - h$, for $k = 1, 2, \ldots$. Since $y_0 = 1$, we have $y_1 = 1 + 2h - h = 1 + h = (1 + 2h)/2 + 1/2$, and hence $y_2 = (1 + 2h)y_1 - h = (1 + 2h)^2/2 + (1 + 2h)/2 - h = (1 + 2h)^2/2 + 1/2$. Furthermore, $y_3 = (1 + 2h)y_2 - h = (1 + 2h)^3/2 + (1 + 2h)/2 - h = (1 + 2h)^3/2 + 1/2$. Continuing in this fashion (or using induction) we obtain $y_k = (1 + 2h)^k/2 + 1/2$. For fixed $t > 0$ choose $h = t/k$. Then substitute for h in the last formula to obtain

$y_k = (1 + 2t/k)^k/2 + 1/2$. Letting $k \to \infty$ we find (see hint for Problem 20(d)) that $y(t) = \lim_{k\to\infty} y_k = e^{2t}/2 + 1/2$, which is the exact solution.

2.8

1. Let $s = t - 1$ and $w(s) = y(t(s)) - 2$, then when $t = 1$ and $y = 2$ we have $s = 0$ and $w(0) = 0$. Also, $dw/ds = (dw/dt)(dt/ds) = (d/dt)(y - 2)(dt/ds) = dy/dt$ (since $t = s + 1$) and hence $dw/ds = (s + 1)^2 + (w + 2)^2$, upon substitution into the given differential equation.

4.(a) The approximating functions are defined recursively by

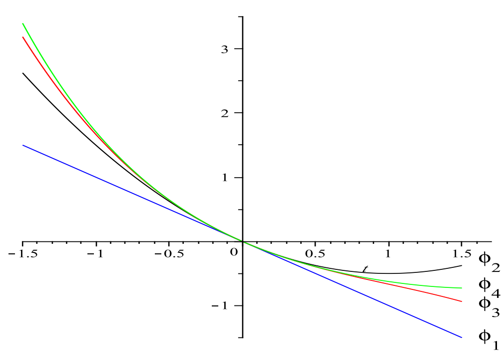
$$\phi_{n+1}(t) = \int_0^t [-\phi_n(s) - 1]\, ds\,.$$

Setting $\phi_0(t) = 0\,,\ \phi_1(t) = -t\,.$ Continuing, $\phi_2(t) = -t + t^2/2$, $\phi_3(t) = -t + t^2/2 - t^3/(2\cdot 3)$, $\phi_4(t) = -t + t^2/2 - t^3/3! + t^4/4!$, ... Based upon these we conjecture that $\phi_n(t) = \sum_{k=1}^{n}(-1)^k t^k/k\,!$ and use mathematical induction to verify this form for $\phi_n(t)$. First, let $n = 1$, then $\phi_n(t) = -t$, so it is certainly true for $n = 1$. Then, using Eq.(7) again we have

$$\phi_{n+1}(t) = \int_0^t [-\phi_n(s) - 1]\, ds = \int_0^t [-\sum_{k=1}^{n} \frac{(-1)^k}{k\,!} s^k - 1]\, ds = \sum_{k=1}^{n+1} \frac{(-1)^k}{k\,!} t^k,$$

and we have verified our conjecture.

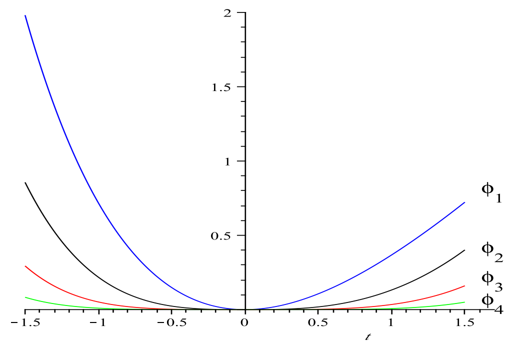
(b)



(c) Recall from calculus that $e^{at} = 1 + \sum_{k=1}^{\infty} a^k t^k/k!$. Thus

$$\phi(t) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k\,!} t^k = e^{-t} - 1\,.$$

(d)



From the plot it appears that ϕ_4 is a good estimate for $|t| < 1$.

7.(a) The approximating functions are defined recursively by

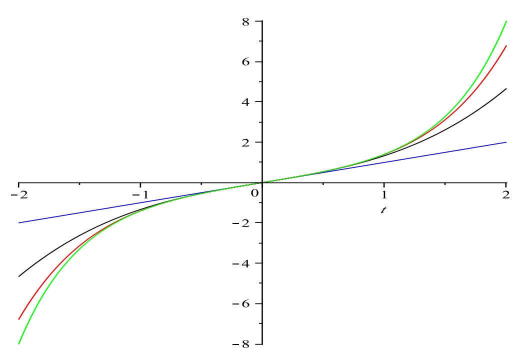
$$\phi_{n+1}(t) = \int_0^t [s\phi_n(s) + 1]\, ds\,.$$

Setting $\phi_0(t) = 0\,,\ \phi_1(t) = t\,.$ Continuing, $\phi_2(t) = t + t^3/3$, $\phi_3(t) = t + t^3/3 + t^5/(3 \cdot 5)$, $\phi_4(t) = t + t^3/3 + t^5/(3 \cdot 5) + t^7/(3 \cdot 5 \cdot 7), \ldots$. Based upon these we conjecture that $\phi_n(t) = \sum_{k=1}^{n} t^{2k-1}/(1 \cdot 3 \cdot 5 \cdots (2k-1))$ and use mathematical induction to verify this form for $\phi_n(t)$. First, let $n = 1$, then $\phi_n(t) = t$, so it is certainly true for $n = 1$. Then, using Eq.(7) again we have

$$\phi_{n+1}(t) = \int_0^t [s\phi_n(s) + 1]\, ds = \int_0^t [\sum_{k=1}^{n} s \frac{s^{2k-1}}{1 \cdot 3 \cdots (2k-1)} + 1]\, ds = \sum_{k=1}^{n+1} \frac{t^{2k-1}}{1 \cdot 3 \cdots (2k-1)},$$

and we have verified our conjecture.

(b)



(c) Using the identity $\phi_n(t) = \phi_1(t) + [\phi_2(t) - \phi_1(t) + [\phi_3(t) - \phi_2(t)] + \ldots + [\phi_n(t) - \phi_{n-1}(t)]$, consider the series $\phi_1(t) + \sum_{k=1}^{\infty}[\phi_{k+1}(t) - \phi_k(t)]$. Fix any t value now.

We use the Ratio Test to prove the convergence of this series:

$$\left|\frac{\phi_{k+1}(t)-\phi_k(t)}{\phi_k(t)-\phi_{k-1}(t)}\right| = \left|\frac{\frac{t^{2k+1}}{1\cdot 3\cdots(2k+1)}}{\frac{t^{2k-1}}{1\cdot 3\cdots(2k-1)}}\right| = \frac{|t|^2}{2k+1}.$$

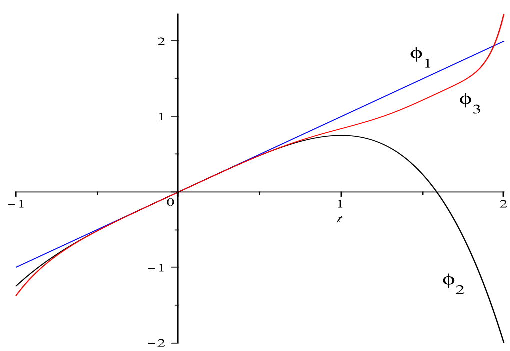
The limit of this quantity is 0 for any fixed t as $k \to \infty$, and we obtain that $\phi_n(t)$ is convergent for any t.

10.(a) The approximating functions are defined recursively by

$$\phi_{n+1}(t) = \int_0^t \left[1 - \phi_n^3(s)\right] ds\,.$$

Set $\phi_0(t) = 0$. The first three iterates are given by $\phi_1(t) = t\,,$ $\phi_2(t) = t - t^4/4\,,$ $\phi_3(t) = t - t^4/4 + 3t^7/28 - 3t^{10}/160 + t^{13}/832$.

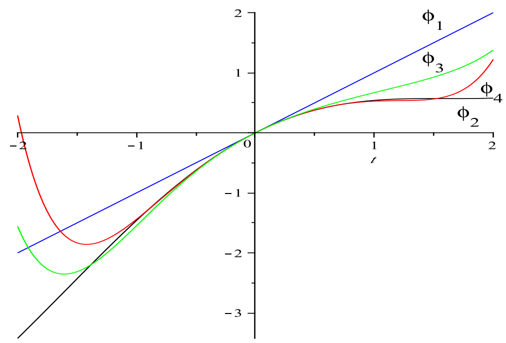
(b)



The approximations appear to be diverging.

11.(a) First, recall that $\sin x = x - x^3/3! + x^5/5! + O(x^7)$. Now, for this problem, $\phi_1(t) = \int_0^t [1 - \sin\phi_0(s)]\,ds = t$ and we obtain that $\phi_2(t) = \int_0^t [1 - \sin s]\,ds = \int_0^t [1 - (s - s^3/3! + s^5/5! + O(s^7)]\,ds = t - t^2/2! + t^4/4! - t^6/6! + O(t^8)$. For ϕ_3 we need to find $\sin(\phi_2(t))$, which is given by $\sin(\phi_2(t)) = \phi_2(t) - \phi_2^3(t)/3! + \phi_2^5(t)/5! + O(t^7) = (t - t^2/2! + t^4/4! - t^6/6!) - (t - t^2/2!)^3/3! + t^5/5! + O(t^7)$, where we have retained only the terms less than $O(t^7)$. Now use this in $\phi_3(t) = \int_0^t [1 - \sin(\phi_2(s))]\,ds$, which gives the desired answer up to $O(t^8)$.

(b)



13. Note that $\phi_n(0) = 0$ and $\phi_n(1) = 1$, for every $n \geq 1$. Let $a \in (0,1)$. Then $\phi_n(a) = a^n$. Clearly, $\lim_{n\to\infty} a^n = 0$. Hence the assertion is true.

2.9

2. Using the given difference equation we have for $n = 0$, $y_1 = y_0/2$, for $n = 1$, $y_2 = 2y_1/3 = y_0/3$; and for $n = 2$, $y_3 = 3y_2/4 = y_0/4$. Thus we guess that $y_n = y_0/(n+1)$, and the given equation the gives $y_{n+1} = (n+1)y_n/(n+2) = y_0/(n+2)$, which, by mathematical induction, verifies $y_n = y_0/(n+1)$ as the solution for all n. $\lim_{n\to\infty} = 0$, as y_0 is constant.

5. Writing the equation for each $n \geq 0$,

$$\begin{aligned}
y_1 &= 0.5\,y_0 + 6 \\
y_2 &= 0.5\,y_1 + 6 = 0.5(0.5\,y_0 + 6) + 6 = (0.5)^2 y_0 + 6 + (0.5)6 \\
y_3 &= 0.5\,y_2 + 6 = 0.5(0.5\,y_1 + 6) + 6 = (0.5)^3 y_0 + 6\left[1 + (0.5) + (0.5)^2\right] \\
&\vdots \\
y_n &= (0.5)^n y_0 + 12\left[1 - (0.5)^n\ \right],
\end{aligned}$$

which follows from Eq.(13) and (14). The sequence is convergent for all y_0, and in fact $y_n \to 12$.

7. Let y_n be the balance at the end of the nth day. Then $y_{n+1} = (1 + r/356)\,y_n$. The solution of this difference equation is $y_n = (1 + r/365)^n\,y_0$, in which y_0 is the initial balance. At the end of one year, the balance is $y_{365} = (1 + r/365)^{365}\,y_0$. Given that $r = .07$, $y_{365} = (1 + r/365)^{365}\,y_0 = 1.0725\,y_0$. Hence the effective annual yield is $(1.0725\,y_0 - y_0)/y_0 = 7.25\%$.

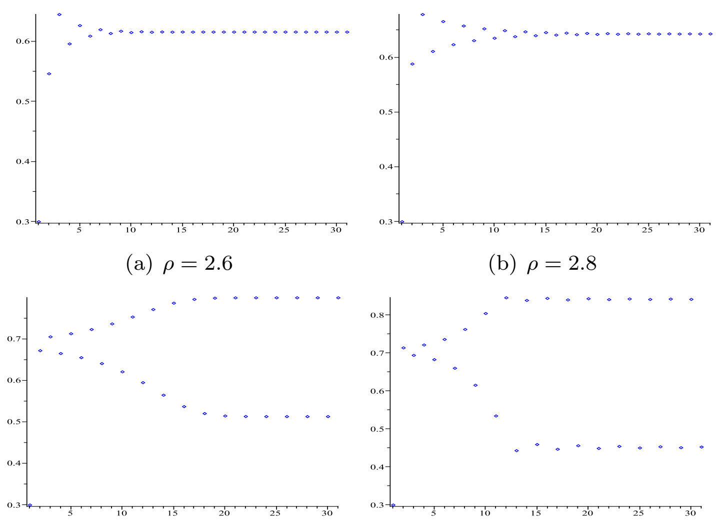
10. As in Ex.(1), the governing equation is $y_{n+1} = \rho y_n - b$, which has the solution $y_n = \rho^n y_0 - (1 - \rho^n)b/(1 - \rho)$. (Eq.(14) with a negative b). Setting $y_{360} = 0$ and solving for b we obtain $b = (1 - \rho)\rho^{360} y_0/(1 - \rho^{360})$, where $\rho = 1.0075$ for part (a).

13. We must solve Eq.(14) numerically for ρ when $n = 240$, $y_{240} = 0$, $b = -\$900$ and $y_0 = \$95,000$. The result is $\rho = 1.0081$, so the monthly interest rate is $r = 0.81\%$, which is equivalent to an annual rate of 9.73%.

14. Substituting Eq.(25), $u_n = (\rho-1)/\rho + v_n$ into Eq.(21) we get $(\rho-1)/\rho + v_{n+1} = \rho((\rho-1)/\rho + v_n)(1-(\rho-1)/\rho - v_n)$, which after simplification turns into $v_{n+1} = -(\rho-1)/\rho + (\rho - 1 + \rho v_n)(1/\rho - v_n) = (1-\rho)/\rho + (\rho-1)/\rho - (\rho-1)v_n + v_n - \rho v_n^2 = (2-\rho)v_n - \rho v_n^2$, which is exactly what we wanted to prove.

15.(a) For $u_0 = 0.2$, we have $u_1 = 3.2u_0(1-u_0) = 0.512$ and $u_2 = 3.2u_1(1-u_1) = 0.7995392$. Likewise, we get $u_3 = 0.51288406$, $u_4 = 0.7994688$, $u_5 = 0.51301899$, $u_6 = 0.7994576$ and $u_7 = 0.5130404$. Continuing, $u_{14} = u_{16} = 0.79945549$ and $u_{15} = u_{17} = 0.51304451$.

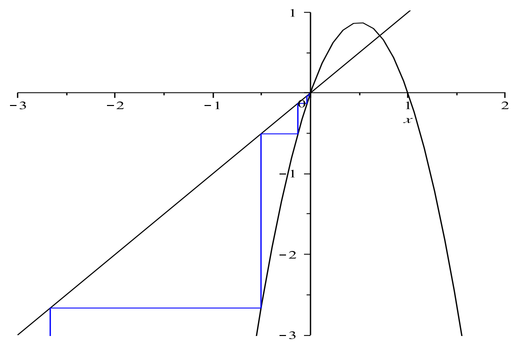
(b) The plots show the nature of solutions.



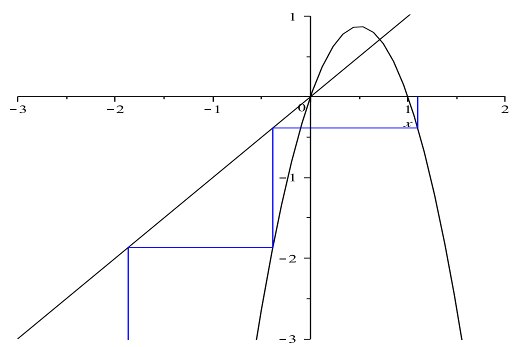
(a) $\rho = 2.6$ (b) $\rho = 2.8$

(c) $\rho = 3.2$ (d) $\rho = 3.4$

16.(a) For example, take $\rho = 3.5$ and $u_0 = -0.01$:



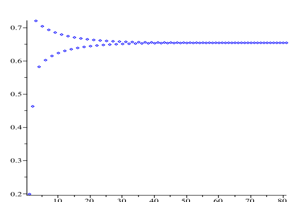
(b) For example, take $\rho = 3.5$ and $u_0 = 1.1$:



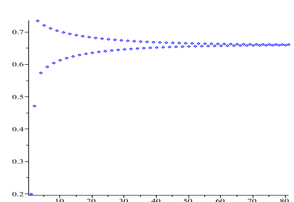
Clearly, $u_n \to -\infty$ as $n \to \infty$.

17. For both parts of this problem a computer was used and an initial value of $u_0 = 0.2$ was chosen. Different initial values or different computer programs may need a slightly different number of iterations to reach the limiting value.

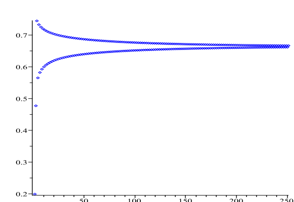
(a)



(a) $\rho = 2.9$



(b) $\rho = 2.95$

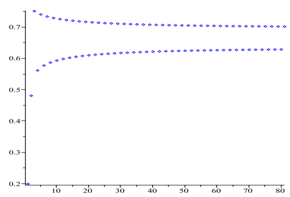


(c) $\rho = 2.99$

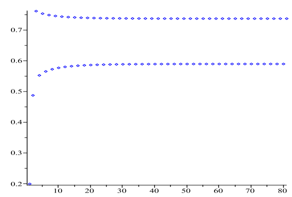
The limiting value of 0.65517 (to 5 decimal places) is reached after approximately 100 iterations for $\rho = 2.9$. The limiting value of 0.66102 (to 5 decimal places) is reached after approximately 200 iterations for $\rho = 2.95$. The limiting value of

0.66555 (to 5 decimal places) is reached after approximately 910 iterations for $\rho = 2.99$.

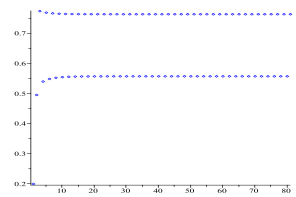
(b)



(a) $\rho = 3.01$



(b) $\rho = 3.05$



(c) $\rho = 3.1$

The solution oscillates between 0.63285 and 0.69938 after approximately 400 iterations for $\rho = 3.01$. The solution oscillates between 0.59016 and 0.73770 after approximately 130 iterations for $\rho = 3.05$. The solution oscillates between 0.55801 and 0.76457 after approximately 30 iterations for $\rho = 3.1$.

18. For an initial value of 0.2 and $\rho = 3.448$ we have the solution oscillating between 0.4403086 and 0.8497146. After approximately 3570 iterations the eighth decimal place is still not fixed, though. For the same initial value and $\rho = 3.45$ the solution oscillates between the four values 0.43399155, 0.84746795, 0.44596778 and 0.85242779 after 3700 iterations. For $\rho = 3.449$ the solution is still varying in the fourth decimal place after 3570 iterations, but there appear to be four values.

PROBLEMS

Before trying to find the solution of a differential equation, it is necessary to know its type. The student should first classify the differential equations before looking at this section, which identifies the type of each differential equation in Problems 1 through 32.

1. The equation is *linear*.

2. The equation is *separable*. Separating the variables gives the differential equation $(2 - \sin y)dy = (1 + \cos x)dx$, and after integration we obtain that the solution is $2y + \cos y - x - \sin x = c$.

3. The equation is *exact*. Simplification gives $(2x + y)dx + (x - 3 - 3y^2)dy = 0$. We can check that $M_y = 1 = N_x$, so the equation is really exact. Integrating M with respect to x gives that $\psi(x, y) = x^2 + xy + g(y)$, then $\psi_y = x + g'(y) = x - 3 - 3y^2$,

which means that $g'(y) = -3 - 3y^2$, so integrating with respect to y we obtain that $g(y) = -3y - y^3$. Therefore the solution is defined implicitly as $x^2 + xy - 3y - y^3 = c$. The initial condition $y(0) = 0$ implies that $c = 0$, so we conclude that the solution is $x^2 + xy - 3y - y^3 = 0$.

4. The equation is *linear*. It can be written as $y' + (2x - 1)y = -3(2x - 1)$, and the integrating factor is e^{x^2-x}. Multiplication by this integrating factor and the subsequent integration gives the solution $ye^{x^2-x} = -3e^{x^2-x} + c$, which means that $y = -3 + ce^{x-x^2}$. (The equation is also *separable*.)

5. The equation is *exact*.

6. The equation is *linear*.

7. The equation is *separable*.

8. The equation is *linear*.

9. The equation is *exact*. Simplification gives $(2xy + 1)dx + (x^2 + 2y)dy = 0$. We can check that $M_y = 2x = N_x$, so the equation is really exact. Integrating M with respect to x gives that $\psi(x, y) = x^2y + x + g(y)$, then $\psi_y = x^2 + g'(y) = x^2 + 2y$, which means that $g'(y) = 2y$, so we obtain that $g(y) = y^2$. Therefore the solution is defined implicitly as $x^2y + x + y^2 = c$.

10. The equation is *separable*. Factoring the terms we obtain the differential equation $(x^2 + x - 1)ydx + x^2(y - 2)dy = 0$. We separate the variables by dividing this equation by yx^2 and obtain

$$(1 + \frac{1}{x} - \frac{1}{x^2})dx + (1 - \frac{2}{y})dy = 0.$$

Integration gives us the solution $x + \ln|x| + 1/x - 2\ln|y| + y = c$. We also have the solution $y = 0$ which we lost when we divided by y.

11. The equation is *exact*.

12. The equation is *linear*. The integrating factor is $\mu(x) = e^{\int dx} = e^x$, which turns the equation into $e^xy' + e^xy = (e^xy)' = e^x/(1 + e^x)$. We can integrate the right hand side by substituting $u = 1 + e^x$, this gives us the solution $ye^x = \ln(1 + e^x) + c$, i.e. $y = ce^{-x} + e^{-x}\ln(1 + e^x)$.

13. The equation is *separable*.

14. The equation is *exact*.

15. The equation is *separable*.

16. The equation is *exact*.

17. The equation is *linear.*

18. The equation is *linear.*

19. The equation is *exact.*

20. The equation is *separable.*

21. The equation is *exact.* Algebraic manipulations give us the symmetric form $(2y^2 + 6xy - 4)dx + (3x^2 + 4xy + 3y^2)dy = 0$. We can check that $M_y = 4y + 6x = N_x$. Integrating M with respect to x gives that $\psi(x, y) = 2y^2x + 3x^2y - 4x + g(y)$, then $\psi_y = 4yx + 3x^2 + g'(y) = 3x^2 + 4xy + 3y^2$, which means that $g'(y) = 3y^2$, so we obtain that $g(y) = y^3$. Therefore the solution is $2xy^2 + 3x^2y - 4x + y^3 = c$.

22. The equation is *separable.*

23. The equation is *linear.*

24. The equation is *exact.*

25. The equation is *exact.*

26. The equation is *homogeneous.* (See Section 2.2, Problem 30) We can write the equation in the form $y' = y/x + e^{y/x}$. We substitute $u(x) = y(x)/x$, which means $y = ux$ and then $y' = u'x + u$. We obtain the equation $u'x + u = u + e^u$, which is a separable equation. Separation of variables gives us $e^{-u}du = (1/x)dx$, so after integration we obtain that $-e^{-u} = \ln|x| + c$ and then substituting $u = y/x$ back into this we get the implicit solution $e^{-y/x} + \ln|x| = c$.

27. The equation can be made *exact* with an appropriate integrating factor. Algebraic manipulations give us the symmetric form $xdx - (x^2y + y^3)dy = 0$. We can check that $(M_y - N_x)/M = 2xy/x = 2y$ depends only on y, which means we will be able to find an integrating factor in the form $\mu(y)$. This integrating factor is $\mu(y) = e^{-\int 2ydy} = e^{-y^2}$. The equation after multiplication becomes

$$e^{-y^2}xdx - e^{-y^2}(x^2y + y^3)dy = 0.$$

This equation is exact now, as we can check that $M_y = -2ye^{-y^2}x = N_x$. Integrating M with respect to x gives that $\psi(x, y) = e^{-y^2}x^2/2 + g(y)$, then $\psi_y = -e^{-y^2}x^2y + g'(y) = -e^{-y^2}(x^2y + y^3)$, which means that $g'(y) = -y^3e^{-y^2}$. We can integrate this expression by substituting $u = -y^2$, $du = -2ydy$. Integrating by parts, we obtain that

$$g(y) = -\int y^3e^{-y^2}dy = -\int \frac{1}{2}ue^u du = -\frac{1}{2}(ue^u - e^u) + c =$$

$$-\frac{1}{2}(-y^2e^{-y^2} - e^{-y^2}) + c.$$

Therefore the solution is defined implicitly as $e^{-y^2}x^2/2 - \frac{1}{2}(-y^2e^{-y^2} - e^{-y^2}) = c$, or (after simplification) as $e^{-y^2}(x^2+y^2+1)=c$. Remark: using the hint and substituting $u=x^2$ gives us $du=2xdx$. The equation turns into $2(uy+y^3)dy=du$, which is a linear equation for u as a function of y. The integrating factor is e^{-y^2} and we obtain the same solution after integration.

28. The equation is *linear* or *homogeneous*; it can also be made *exact* by choosing an appropriate integrating factor.

29. The equation is *homogeneous*.

30. The equation is *homogeneous*.

31. The equation can be made *exact* by choosing an appropriate integrating factor.

32. This is a *Bernoulli* equation. (See Section 2.4, Problem 27) If we substitute $u=y^{-1}$, then $u'=-y^{-2}y'$, so $y'=-u'y^2=-u'/u^2$ and the equation becomes $-xu'/u^2+(1/u)-e^{2x}/u^2=0$, and then $u'-u/x=-e^{2x}/x$, which is a linear equation. The integrating factor is $e^{-\int(1/x)dx}=e^{-\ln x}=1/x$, and we obtain that $(u'/x)-(u/x^2)=(u/x)'=-e^{2x}/x^2$. The integral of the function on the right hand side can not be expressed in a closed form using elementary functions, so we have to express the solution using integrals. Let us integrate both sides of this equation from 1 to x. We obtain that the left hand side turns into

$$\int_1^x (u(s)/s)'ds=(u(x)/x)-(u(1)/1)=\frac{1}{yx}-\frac{1}{y(1)}=\frac{1}{yx}-1/2.$$

The right hand side gives us $-\int_1^x[e^{2s}/s^2]\,ds$. So we find that

$$1/y=-x\int_1^x[e^{2s}/s^2]\,ds+(x/2).$$

34.(a) Using the idea of Problem 33, we obtain that $y=t+(1/v)$, and v satisfies the differential equation $v'=-1$. This means that $v=-t+c$ and then $y=t+(c-t)^{-1}$.

(b) Using the idea of Problem 33, we set $y=(1/t)+(1/v)$, and then v satisfies the differential equation $v'=-1-(v/t)$. This is a linear equation with integrating factor $\mu(t)=t$, and the equation turns into $tv'+v=(tv)'=-t$, which means that $tv=-t^2/2+c$, so $v=-(t/2)+(c/t)$ and $y=(1/t)+(1/v)=(1/t)+2t/(2c-t^2)$.

(c) Using the idea of Problem 33, we set $y=\sin t+(1/v)$. Then v satisfies the differential equation $v'=-\tan t v-1/(2\cos t)$. This is a linear equation with integrating factor $\mu(t)=1/\cos t$, which turns the equation into

$$v'/\cos t+v\sin t/\cos^2 t=(v/\cos t)'=-1/(2\cos^2 t).$$

Integrating this we obtain that $v=c\cos t-(1/2)\sin t$, and the solution is $y=\sin t+(c\cos t-(1/2)\sin t)^{-1}$.

42. Set $y' = v(y)$. Then $y'' = v'(y)(dy/dt) = v'(y)v(y)$. The equation turns into $yv'v + v^2 = 0$, where the differentiation is with respect to y now. This is a separable equation, separation of variables yields $-dv/v = dy/y$, and then $-\ln v = \ln y + \tilde{c}$, so $v = 1/(cy)$. Now this implies that $y' = 1/(cy)$, where the differentiation is with respect to t. This is another separable equation and we obtain that $cydy = 1dt$, so $cy^2/2 = t + d$ and the solution is defined implicitly as $y^2 = c_1 t + c_2$.

45. Set $y' = v(y)$. Then $y'' = v'(y)(dy/dt) = v'(y)v(y)$. We obtain the equation $2y^2v'v + 2yv^2 = 1$, where the differentiation is with respect to y. This is a *Bernoulli* equation (See Section 2.4, Problem 27) and substituting $z = v^2$ we get that $z' = 2vv'$, which means that the equation reads $y^2z' + 2yz = (y^2z)' = 1$. Integration yields $v^2 = z = (1/y) + (c/y^2)$, so $y' = v = \pm\sqrt{y+c}/y$. This is a separable equation; separating the variables we get $\pm ydy/\sqrt{y+c} = dt$ and then the implicitly defined solution is obtained by integration: $\pm((2/3)(y+c)^{3/2} - 2c(y+c)^{1/2}) = t + d$.

47. Set $y' = v(y)$. Then $y'' = v'(y)(dy/dt) = v'(y)v(y)$. We obtain the equation $v'v + v^2 = 2e^{-y}$, where the differentiation is with respect to y. This is a *Bernoulli* equation (See Section 2.4, Problem 27) and substituting $z = v^2$ we get that $z' = 2vv'$, which means that the equation reads $z' + 2z = 4e^{-y}$. The integrating factor is $\mu(y) = e^{2y}$, which turns the equation into $e^{2y}z' + 2e^{2y}z = (e^{2y}z)' = 4e^y$. Integration gives us $v^2 = z = 4e^{-y} + ce^{-2y}$. This implies that $y' = v = \pm e^{-y}\sqrt{c + 4e^y}$. Separation of variables now shows that $\pm e^y dy/\sqrt{c+4e^y} = dt$. Integration and simplification gives $\pm(1/2)(c + 4e^y)^{1/2} = t + d$. Algebraic manipulations then yield the implicitly defined solution $e^y = (t + c_2)^2 + c_1$.

48. Suppose that $y' = v(y)$ and then $y'' = v'(y)v(y)$. The equation is $v^2v' = 2$, which gives us $v^3/3 = 2y + c$. Now plugging 0 in place of t gives that $2^3/3 = 2 \cdot 1 + c$ and we get that $c = 2/3$. This turns into $v^3 = 6y + 2$, i.e. $y' = (6y+2)^{1/3}$. This separable equation gives us $(6y+2)^{-1/3}dy = dt$, and integration shows that $(1/6)(3/2)(6y+2)^{2/3} = t + d$. Again, plugging in $t = 0$ gives us $d = 1$ and the solution is $(6y+2)^{2/3} = 4(t+1)$. Solving for y here yields $y = (4/3)(t+1)^{3/2} - 1/3$.

51. Set $v = y'$, then $v' = y''$. The equation with this substitution is $vv' = t$. Integrating this separable differential equation we get that $v^2/2 = t^2/2 + c$, and $c = 0$ from the initial conditions. This implies that $y' = v = t$, so $y = t^2/2 + d$, and the initial conditions again imply that the solution is $y = t^2/2 + 3/2$.