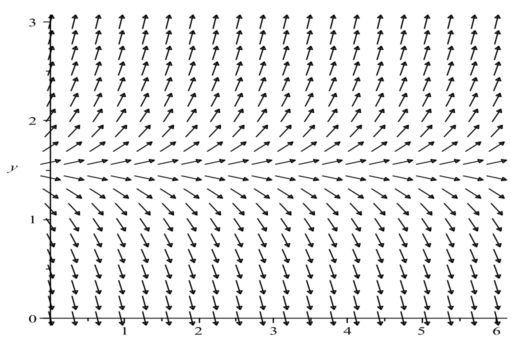
CHAPTER

1

Introduction

1.1

2.



For $y > 3/2$, the slopes are positive, therefore the solutions increase. For $y < 3/2$, the slopes are negative, therefore, the solutions decrease. As a result, y diverges from $3/2$ as $t \to \infty$ if $y(0) \neq 3/2$.

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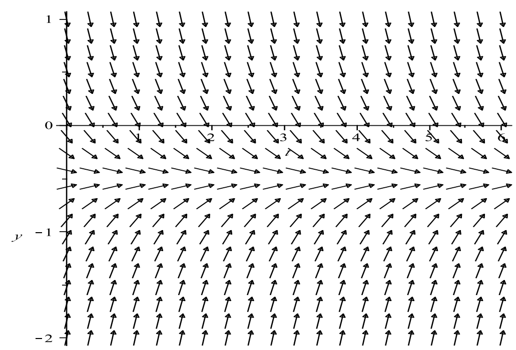
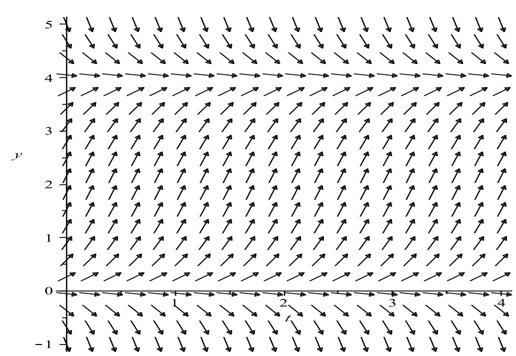


For $y > -1/2$, the slopes are negative, therefore the solutions decrease. For $y < -1/2$, the slopes are positive, therefore, the solutions increase. As a result, $y \to -1/2$ as $t \to \infty$.

7. For the solutions to satisfy $y \to 3$ as $t \to \infty$, we need $y' < 0$ for $y > 3$ and $y' > 0$ for $y < 3$. The equation $y' = 3 - y$ satisfies these conditions. (This is not unique as there are other possible answers, such as $y' = 6 - 2y$.)

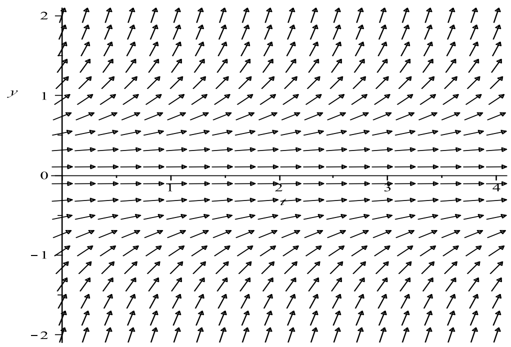
9. For solutions other than $y(t) = 2$ to diverge from $y = 2$, $y(t)$ must be an increasing function for $y > 2$, and a decreasing function for $y < 2$. The simplest differential equation whose solutions satisfy these criteria is $y' = y - 2$.

11.



For $y = 0$ and $y = 4$ we have $y' = 0$ and thus $y = 0$ and $y = 4$ are equilibrium solutions. For $y > 4$, $y' < 0$ so if $y(0) > 4$ the solution approaches $y = 4$ from above. If $0 < y(0) < 4$, then $y' > 0$ and the solutions "grow" to $y = 4$ as $t \to \infty$. For $y(0) < 0$ we see that $y' < 0$ and the solutions diverge from 0.

13.



Since $y' = y^2$, $y = 0$ is the only equilibrium solution and $y' > 0$ for all y. Thus $y \to 0$ if the initial value is negative; y diverges from 0 if the initial value is positive.

16. From Figure 1.1.6 we can see that $y = 2$ is an equilibrium solution and thus (c) and (j) are the only possible differential equations to consider. Since $dy/dt > 0$ for $y > 2$, and $dy/dt < 0$ for $y < 2$ we conclude that (c) is the correct answer: $y' = y - 2$.

19. From Figure 1.1.9 we can see that $y = 0$ and $y = 3$ are equilibrium solutions, so (e) and (h) are the only possible differential equations. Furthermore, we have $dy/dt < 0$ for $y > 3$ and for $y < 0$, and $dy/dt > 0$ for $0 < y < 3$. This tells us that (h) is the desired differential equation: $y' = y\,(3 - y)$.

21.(a) Let $q(t)$ denote the amount of chemical in the pond at time t. The chemical q will be measured in grams and the time t will be measured in hours. The rate at which the chemical is entering the pond is given by 300 gallons/hour $\cdot .01$ grams/gallons $= 300 \cdot 10^{-2}$ grams/hour. The rate at which the chemical leaves the pond is given by 300 gallons/hour $\cdot q/1,000,000$ grams/gallons $= 300 \cdot q10^{-6}$ grams/hour. Thus the differential equation is given by $dq/dt = 300(10^{-2} - q10^{-6})$.

(b) The equilibrium solution occurs when $q' = 0$, or $q = 10^4$ grams. Since $q' > 0$ for $q < 10^4$ gm and $q' < 0$ for $q > 10^4$ gm, all solutions approach the equilibrium solution independent of the amount present at $t = 0$.

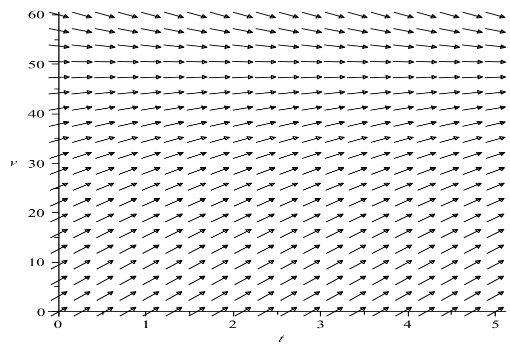
22. The surface area of a spherical raindrop of radius r is given by $S = 4\pi r^2$. The volume of a spherical raindrop is given by $V = 4\pi r^3/3$. Therefore, we see that the surface area $S = cV^{2/3}$ for some constant c. If the raindrop evaporates at a rate proportional to its surface area, then $dV/dt = -kV^{2/3}$ for some $k > 0$.

25.(a) Following the discussion in the text, the differential equation is $m(dv/dt) = mg - \gamma\, v^2$, or equivalently, $dv/dt = g - \gamma v^2/m$.

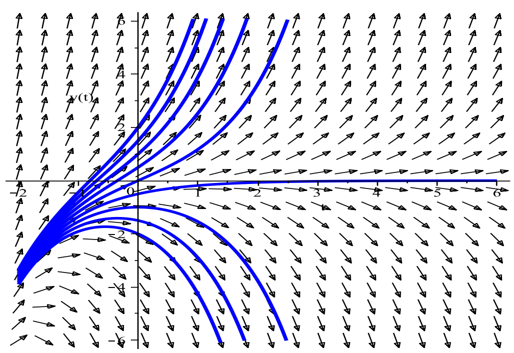
(b) After a long time, $dv/dt \approx 0$. Hence the object attains a terminal velocity given by $v_\infty = \sqrt{mg/\gamma}\,$.

(c) Using the relation $\gamma\, v_\infty^2 = mg$, the required drag coefficient is $\gamma = 2/49$ kg/s.

(d)

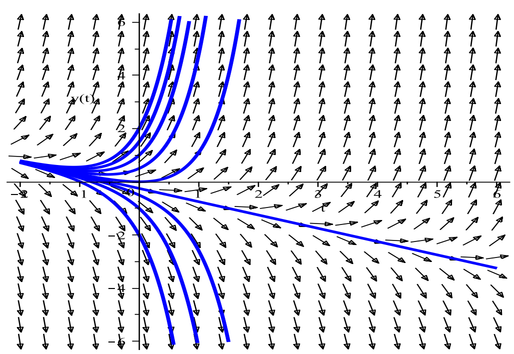


28.



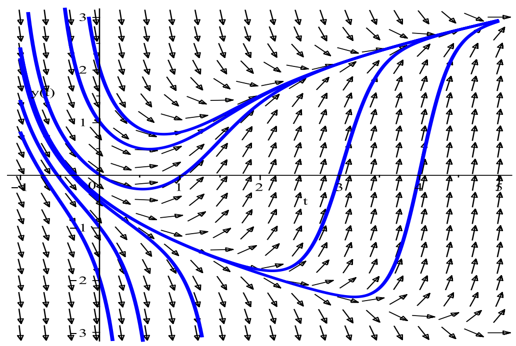
Solutions appear to diverge from $y = 0$.

29.



All solutions (except $y(0) = -1/4$) diverge from the solution $y(t) = -t/2 - 1/4$ and approach $\pm\infty$.

31.



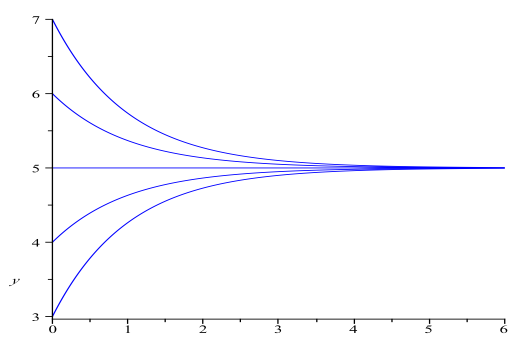
Solutions approach $-\infty$ or are asymptotic to $\sqrt{2t-1}$.

1.2

1.(a) The differential equation can be rewritten as

$$\frac{dy}{5-y} = dt\,.$$

Integrating both sides of this equation results in $-\ln|5-y| = t + c_1$, or equivalently, $5 - y = c\,e^{-t}$. Applying the initial condition $y(0) = y_0$ results in the specification of the constant as $c = 5 - y_0$. Hence the solution is $y(t) = 5 + (y_0 - 5)e^{-t}$.

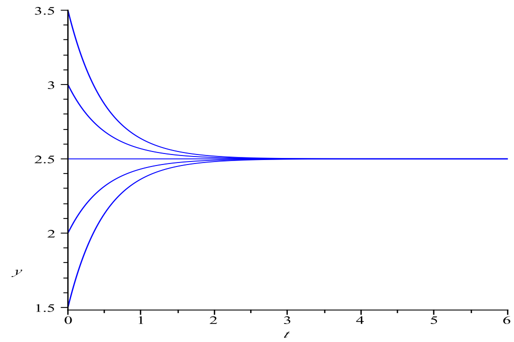


All solutions appear to converge to the equilibrium solution $y(t) = 5$.

(b) The differential equation can be rewritten as

$$\frac{dy}{5-2y} = dt\,.$$

Integrating both sides of this equation results in $-(1/2)\ln|5-2y| = t + c_1$, or equivalently, $5 - 2y = c\,e^{-2t}$. Applying the initial condition $y(0) = y_0$ results in the specification of the constant as $c = 5 - 2y_0$. Hence $y(t) = 5/2 + (y_0 - 5/2)e^{-2t}$.

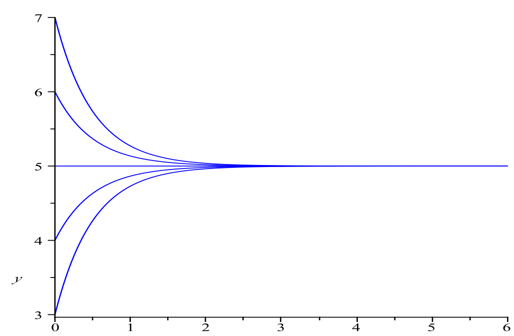


All solutions appear to converge to the equilibrium solution $y(t) = 5/2$, at a faster rate than in part (a).

(c) Rewrite the differential equation as

$$\frac{dy}{10 - 2y} = dt\,.$$

Integrating both sides of this equation results in $-(1/2)\ln|10 - 2y| = t + c_1$, or equivalently, $5 - y = c\,e^{-2t}$. Applying the initial condition $y(0) = y_0$ results in the specification of the constant as $c = 5 - y_0$. Hence $y(t) = 5 + (y_0 - 5)e^{-2t}$.

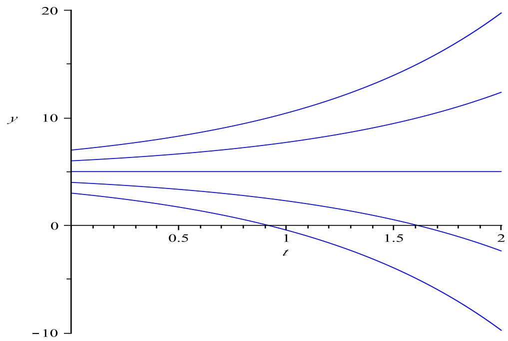


All solutions appear to converge to the equilibrium solution $y(t) = 5$, at a faster rate than in part (a), and at the same rate as in part (b).

2.(a) The differential equation can be rewritten as

$$\frac{dy}{y - 5} = dt\,.$$

Integrating both sides of this equation results in $\ln|y - 5| = t + c_1$, or equivalently, $y - 5 = c\,e^{t}$. Applying the initial condition $y(0) = y_0$ results in the specification of the constant as $c = y_0 - 5$. Hence the solution is $y(t) = 5 + (y_0 - 5)e^{t}$.

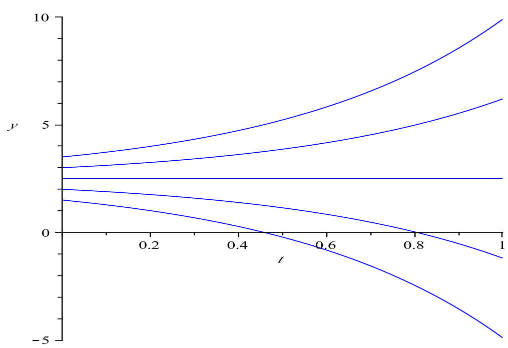


All solutions appear to diverge from the equilibrium solution $y(t) = 5$.

(b) Rewrite the differential equation as

$$\frac{dy}{2y-5} = dt\,.$$

Integrating both sides of this equation results in $(1/2)\ln|2y-5| = t + c_1$, or equivalently, $2y - 5 = c\,e^{2t}$. Applying the initial condition $y(0) = y_0$ results in the specification of the constant as $c = 2y_0 - 5$. So the solution is $y(t) = (y_0 - 5/2)e^{2t} + 5/2$.

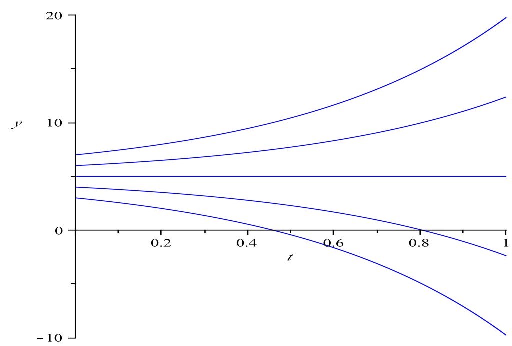


All solutions appear to diverge from the equilibrium solution $y(t) = 5/2$.

(c) The differential equation can be rewritten as

$$\frac{dy}{2y-10} = dt\,.$$

Integrating both sides of this equation results in $(1/2)\ln|2y-10| = t + c_1$, or equivalently, $y - 5 = c\,e^{2t}$. Applying the initial condition $y(0) = y_0$ results in the specification of the constant as $c = y_0 - 5$. Hence the solution is $y(t) = 5 + (y_0 - 5)e^{2t}$.



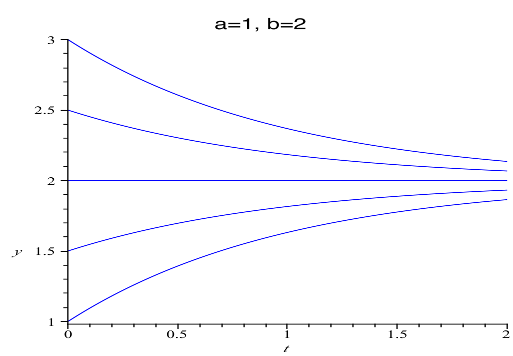
All solutions appear to diverge from the equilibrium solution $y(t) = 5$.

3.(a) Rewrite the differential equation as

$$\frac{dy}{b - ay} = dt\,,$$

which is valid for $y \neq b\,/a$. Integrating both sides results in $-(1/a)\ln|b - ay| = t + c_1$, or equivalently, $b - ay = c\,e^{-at}$. Hence the general solution is $y(t) = (b - c\,e^{-at})/a$. Note that if $y = b/a$, then $dy/dt = 0$, and $y(t) = b/a$ is an equilibrium solution.

(b)



(c) (i) As a increases, the equilibrium solution gets closer to $y(t) = 0$, from above. The convergence rate of all solutions is a. As a increases, the solutions converge to the equilibrium solution quicker.

(ii) As b increases, then the equilibrium solution $y(t) = b/a$ also becomes larger. In this case, the convergence rate remains the same.

(iii) If a and b both increase but b/a =constant, then the equilibrium solution $y(t) = b/a$ remains the same, but the convergence rate of all solutions increases.

5.(a) Rewrite Eq.(ii) as $(1/y)dy/dt = a$ and thus $\ln|y| = at + C$, or $y_1 = ce^{at}$.

(b) If $y = y_1(t) + k$, then $dy/dt = dy_1/dt$. Substituting both these into Eq.(i) we get $dy_1/dt = a(y_1 + k) - b$. Since $dy_1/dt = ay_1$, this leaves $ak - b = 0$ and thus

$k = b/a$. Hence $y = y_1(t) + b/a$ is the solution to Eq(i).

(c) Substitution of $y_1 = ce^{at}$ shows this is the same as that given in Eq.(17).

7.(a) The general solution is $p(t) = 900 + c\,e^{t/2}$, that is, $p(t) = 900 + (p_0 - 900)e^{t/2}$. With $p_0 = 850$, the specific solution becomes $p(t) = 900 - 50e^{t/2}$. This solution is a decreasing exponential, and hence the time of extinction is equal to the number of months it takes, say t_f, for the population to reach zero. Solving $900 - 50e^{t_f/2} = 0$, we find that $t_f = 2\,\ln(900/50) \approx 5.78$ months.

(b) The solution, $p(t) = 900 + (p_0 - 900)e^{t/2}$, is a decreasing exponential as long as $p_0 < 900$. Hence $900 + (p_0 - 900)e^{t_f/2} = 0$ has only one root, given by

$$t_f = 2\,\ln\left(\frac{900}{900 - p_0}\right) \text{ months.}$$

(c) The answer in part (b) is a general equation relating time of extinction to the value of the initial population. Setting $t_f = 12$ months, the equation may be written as $900/(900 - p_0) = e^6$, which has solution $p_0 \approx 897.8$. Since p_0 is the initial population, the appropriate answer is $p_0 = 898$ mice.

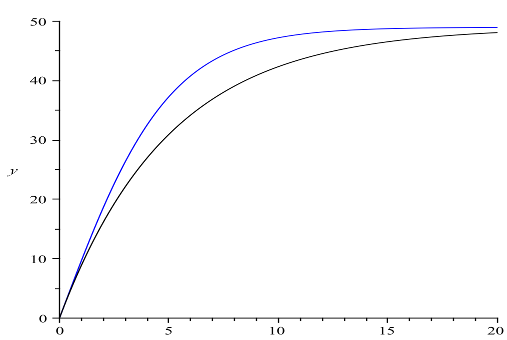
9.(a) The solution to this problem is given by Eq.(26), which has a limiting velocity of 49 m/s. Substituting $v = 48.02$ (which is 98% of 49) into Eq.(26) yields $e^{t/5} = 0.02$. Solving for t we have $t = -5\ln(0.02) = 19.56$ s.

(b) Use Eq.(29) with $t = 19.56$.

11.(a) If the drag force is proportional to v^2 then $F = 98 - kv^2$ is the net force acting on the falling mass ($m = 10$ kg). Thus $10dv/dt = 98 - kv^2$, which has a limiting velocity of $v^2 = 98/k$. Setting $v^2 = 49^2$ gives $k = 98/49^2$ and hence $dv/dt = (49^2 - v^2)/(10/k) = (49^2 - v^2)/245$.

(b) From part (a) we have $dv/(49^2 - v^2) = (1/245)dt$, which after integration yields $(1/49)\tanh^{-1}(v/49) = t/245 + C_0$. Setting $t = 0$ and $v = 0$, we have $0 = 0 + C_0$, or $C_0 = 0$. Thus $\tanh^{-1}(v/49) = t/5$, or $v(t) = 49\tanh(t/5)$ m/s.

(c)



(d) In the graph of part (c), the solution for the linear drag force lies below the solution for the quadratic drag force. This latter solution approaches equilibrium faster since, as the velocity increases, there is a larger drag force.

(e) Note that $\int \tanh(x)\,dx = \ln(\cosh x) + C$. The distance the object falls is given by $x = 245\ln(\cosh(t/5))$.

(f) Solving $300 = 245\ln(\cosh(T/5))$ gives $T \approx 9.48$ s.

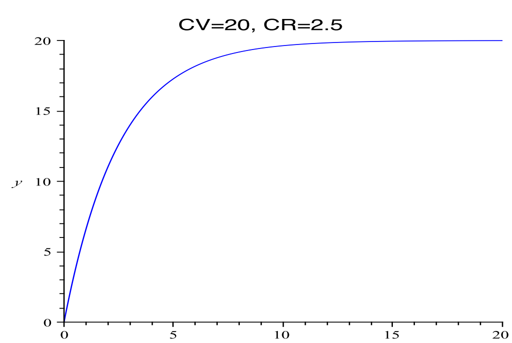
12.(a,b) The general solution of the differential equation is $Q(t) = c\,e^{-rt}$. Given that $Q(0) = 100$ mg, the value of the constant is given by $c = 100$. Hence the amount of thorium-234 present at any time is given by $Q(t) = 100\,e^{-rt}$. Furthermore, based on the hypothesis, setting $t = 1$ results in $82.04 = 100\,e^{-r}$. Solving for the rate constant, we find that $r = -\ln(82.04/100) \approx .19796$/week or $r \approx .02828$/day.

(c) Let T be the time that it takes the isotope to decay to one-half of its original amount. From part (a), it follows that $50 = 100\,e^{-rT}$, in which $r = .19796$/week. Taking the natural logarithm of both sides, we find that $T \approx 3.5014$ weeks or $T \approx 24.51$ days.

15.(a) Rewrite the differential equation as $du/(u-T) = -kdt$ and then integrate to find $\ln|u-T| = -kt + c$. Thus $u - T = \pm Ce^{-kt}$. For $t = 0$ we have $u_0 - T = \pm C$ and thus $u(t) = T + (u_0 - T)e^{-kt}$.

(b) Set $u(\tau) - T = (u_0 - T)/2$ in the solution in part (a). We obtain $1/2 = e^{-k\tau}$, or $k\tau = \ln 2$.

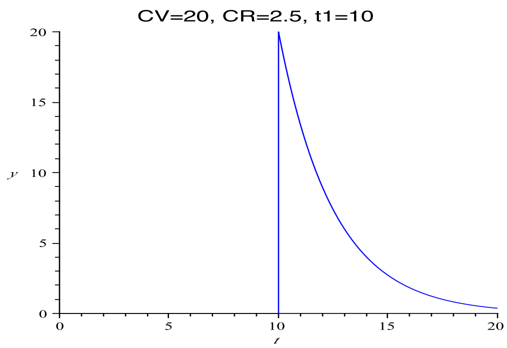
17.(a) Rewrite the differential equation as $dQ/(Q - CV) = -(1/CR)dt$, thus, upon integrating and simplifying, we get $Q = De^{-t/CR} + CV$. $Q(0) = 0$ implies that the solution of the differential equation is $Q(t) = CV(1 - e^{-t/CR})$.



(b) As $t \to \infty$, the exponential term vanishes, and the limiting value is $Q_L = CV$.

(c) In this case $RdQ/dt + Q/C = 0$. $Q(t_1) = CV$. The solution of this differential equation is $Q(t) = Ee^{-t/CR}$, so $Q(t_1) = Ee^{-t_1/CR} = CV$, or $E = CVe^{t_1/CR}$. Thus

$Q(t) = CVe^{t_1/CR}e^{-t/CR} = CVe^{-(t-t_1)/CR}$ for $t \geq t_1$.



1.3

2. The differential equation is second order since the highest derivative in the equation is of order two. The equation is nonlinear due to the y^2 term (as well as due to the y^2 term multiplying the y'' term).

6. The differential equation is third order since the highest derivative in the equation is of order three. The equation is linear because the left hand side is a linear function of y and its derivatives, and the right hand side is only a function of t.

8. $y_1 = e^{-3t}$, so $y_1' = -3e^{-3t}$ and $y_1'' = 9e^{-3t}$. This implies that $y_1'' + 2y_1' - 3y_1 = (9 - 6 - 3)e^{-3t} = 0$. Also, $y_2 = e^t$, so $y_2' = y_2'' = e^t$. This gives $y_2'' + 2y_2' - 3y_2 = (1 + 2 - 3)e^t = 0$.

11. $y_1(t) = t^{1/2}$, so $y_1'(t) = t^{-1/2}/2$ and $y_1''(t) = -t^{-3/2}/4$. Substituting into the left hand side of the equation, we have $2t^2(-t^{-3/2}/4) + 3t(t^{-1/2}/2) - t^{1/2} = -t^{1/2}/2 + 3\,t^{1/2}/2 - t^{1/2} = 0$. Likewise, $y_2(t) = t^{-1}$, so $y_2'(t) = -t^{-2}$ and $y_2''(t) = 2\,t^{-3}$. Substituting into the left hand side of the differential equation, we have $2t^2(2\,t^{-3}) + 3t(-t^{-2}) - t^{-1} = 4\,t^{-1} - 3\,t^{-1} - t^{-1} = 0$. Hence both functions are solutions of the differential equation.

14. Recall that if $u(t) = \int_0^t f(s)\,ds$, then $u'(t) = f(t)$. Now $y = e^{t^2}\int_0^t e^{-s^2}\,ds + e^{t^2}$, so $y' = 2te^{t^2}\int_0^t e^{-s^2}\,ds + 1 + 2te^{t^2}$. Therefore, $y' - 2ty = 2te^{t^2}\int_0^t e^{-s^2}\,ds + 1 + 2te^{t^2} - 2t(e^{t^2}\int_0^t e^{-s^2}\,ds + e^{t^2}) = 1$.

16. Let $y(t) = e^{rt}$. Then $y''(t) = r^2e^{rt}$, and substitution into the differential equation results in $r^2e^{rt} - e^{rt} = 0$. Since $e^{rt} \neq 0$, we obtain the algebraic equation $r^2 - 1 = 0$. The roots of this equation are $r_{1,2} = \pm 1$.

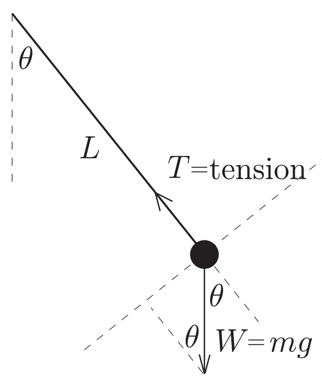
19. Let $y = t^r$. Then $y' = rt^{r-1}$ and $y'' = r(r-1)t^{r-2}$. Substituting these terms into the differential equation, $t^2y'' + 4ty' + 2y = t^2(r(r-1)t^{r-2}) + 4t(rt^{r-1}) + 2t^r =$

$(r(r-1)+4r+2)t^r = 0$. If this is to hold for all t, then we need $r(r-1)+4r+2=0$. Simplifying this expression, we need $r^2+3r+2=0$. The solutions of this equation are $r=-1,-2$.

22. The differential equation is second order since there are second partial derivatives of $u(x,y)$. The differential equation is nonlinear due to the product of $u(x,y)$ times u_x (or u_y).

26. Since $\partial u_1/\partial t = -\alpha^2 e^{-\alpha^2 t}\sin x$ and $\partial^2 u_1/\partial x^2 = -e^{-\alpha^2 t}\sin x$ we have $\alpha^2 u_{xx} = u_t$, for all t and x.

29.(a)



(b) The path of the particle is a circle, therefore polar coordinates are intrinsic to the problem. The variable r is radial distance and the angle θ is measured from the vertical.

Newton's Second Law states that $\sum F = ma$. In the tangential direction, the equation of motion may be expressed as $\sum F_\theta = m\,a_\theta$, in which the tangential acceleration, that is, the linear acceleration along the path is $a_\theta = L\,d^2\theta/dt^2$. (a_θ is positive in the direction of increasing θ). Since the only force acting in the tangential direction is the component of weight, the equation of motion is

$$-mg\,\sin\,\theta = mL\frac{d^2\theta}{dt^2}\,.$$

(c) Rearranging the terms results in the differential equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\,\theta = 0\,.$$